EXTENSIONS OF THE UNIVERSAL THETA DIVISOR

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Abstract. The Jacobian varieties of smooth curves fit together to form a family, the universal Jacobian, over the moduli space of smooth marked curves, and the theta divisors of these curves form a divisor in the universal Jacobian. In this paper we describe how to extend these families over the moduli space of stable marked curves using a stability parameter. We then prove a wall-crossing formula describing how the theta divisor varies with the stability parameter.

We use this result to analyze a divisor on the moduli space of smooth marked curves that has recently been studied by Grushevsky–Zakharov, Hain and Müller. Finally, we compute the pullback of the theta divisor studied in Alexeev's work on stable abelic varieties and in Caporaso's work on theta divisors of compactified Jacobians.

1. Introduction

In this paper we describe how the theta divisor of a compactified universal Jacobian varies with a stability parameter and then use this result to analyze a divisor on the moduli space of curves recently studied by Samuel Grushevsky, Richard Hain, Fabian Müller, and Dmitry Zakharov, and we begin by recalling their work.

Given a sequence $\vec{d} = (d_1, \dots, d_n)$ of integers with $\sum d_j = g-1$ and at least one d_j negative, the subset

$$D_{\vec{d}} := \{(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n} : h^0(C, \mathcal{O}(d_1 p_1 + \dots + d_n p_n)) \neq 0\}$$

is a proper closed subset of $\mathcal{M}_{g,n}$, so it has an associated fundamental class $[D_{\vec{d}}] \in \mathrm{A}^1(\mathcal{M}_{g,n})$, and we can consider the problem of extending $[D_{\vec{d}}]$ to a Chow class $[\overline{D}_{\vec{d}}] \in \mathrm{A}^1(\overline{\mathcal{M}}_{g,n})$ on the Deligne–Mumford compactification, and then describing $[\overline{D}_{\vec{d}}]$ in terms of standard generators. Müller extended $D_{\vec{d}}$ to its Zariski closure $\overline{D}_{\vec{d}}(\text{Mü})$ and proved [Mül13, Theorem 5.6]:

$$[\overline{D}_{\vec{d}}(\text{Mü})] = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j - \sum_{\substack{i,S \\ S \subseteq S^+}} \binom{|d_S - i| + 1}{2} \cdot \delta_{i,S} - \sum_{\substack{i,S \\ S \not\subseteq S^+}} \binom{d_S - i + 1}{2} \cdot \delta_{i,S}. \tag{1}$$

Here $d_S := \sum_{j \in S} d_j$ and $S^+ := \{j \in \{1, \dots, n\} : d_j > 0\}$.

Hain extended $[D_{\vec{d}}]$ to a rational Chow class $[\overline{D}_{\vec{d}}(\text{Ha})]$ using the formalism of theta functions and then proved [Hai13, Theorem 11.7]:

$$[\overline{D}_{\vec{d}}(\text{Ha})] = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j - \sum_{i,S} \binom{d_S - i + 1}{2} \cdot \delta_{i,S} + \frac{\delta_{irr}}{8}. \tag{2}$$

Using different methods, both results were reproved by Grushevsky and Zakharov [GZ14, Theorem 2, Theorem 6].

A third way of extending $[D_{\vec{d}}]$ was suggested by Hain [Hai13, Section 11.2, page 561]. If $\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}$ is the family of degree $g-1$ Jacobians associated to the universal curve over $\mathcal{M}_{g,n}$ (so the fiber of $\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}$ over $(C, p_1, \ldots, p_n)$ is the moduli scheme of degree $g-1$ line bundles on C), then the rule $(C, p_1, \ldots, p_n) \mapsto \mathcal{O}(d_1 p_1 + \cdots + d_n p_n)$ defines a morphism

$$s_{\vec{d}} \colon \mathcal{M}_{g,n} \to \mathcal{J}_{g,n} \tag{3}$$

with the property that $D_{\vec{d}}$ is the preimage of the theta divisor

$$\Theta := \{(C, p_1, \ldots, p_n; F) \colon h^0(C, F) \neq 0\}.$$

Thus one way to extend $D_{\vec{d}}$ is to extend (3) to a morphism

$$s_{\vec{d}} \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{J}}_{g,n}, \tag{4}$$

into an extension $\overline{\mathcal{J}}_{g,n}$ of $\mathcal{J}_{g,n}$, to extend the theta divisor to a divisor $\overline{\Theta}$ on $\overline{\mathcal{J}}_{g,n}$, and then to take the preimage $s_{\vec{d}}^{-1}([\overline{\Theta}])$. The difficulty in carrying out this idea is that the obvious extension of $\mathcal{J}_{g,n}$ is badly behaved. The family $\widetilde{\mathcal{J}} \to \overline{\mathcal{M}}_{g,n}$ of moduli spaces of degree $g-1$ line bundles on stable marked curves exists, but it fails to be separated. In particular, $s_{\vec{d}}$ does extend to a morphism into $\widetilde{\mathcal{J}}$, but there is not a *unique* extension, an issue already observed by Hain, who remarks that this is a "subtle problem' [Hai13, Section 11.2, page 561].

One way to extend $\mathcal{J}_{g,n}$ is to use the theory of degenerate principally polarized abelian varieties. The family $(\mathcal{J}_{g,n}/\mathcal{M}_{g,n}, \Theta)$ is a family of principally polarized torsors for abelian varieties, and this family uniquely extends to a family $(\overline{\mathcal{J}}_{g,n}/\overline{\mathcal{M}}_{g,n}, \overline{\Theta})$ of stable semiabelic pairs, or stable principally polarized degenerate abelian varieties. The morphism $s_{\vec{d}}$ is a rational map into $\overline{\mathcal{J}}_{g,n}$, and we can use it extend $[D_{\vec{d}}]$ as

$$[\overline{D}_{\vec{d}}(\mathrm{SP})] := s_{\vec{d}}^{-1}([\overline{\Theta}]).$$

An alternative approach, the focus of the present paper, is to extend $\mathcal{J}_{g,n}$ as moduli space of sheaves. The failure for $\widetilde{J}_{g,n}$ to be separated is intimately related to an invariant of a line bundle on a reducible curve: the multidegree. The multidegree $\deg(F)$ of a line bundle F is defined to be the vector whose components are the degrees of the restrictions of F to the irreducible components of C. To have a well-behaved moduli space of line bundles, one typically imposes a numerical condition on the multidegree of a line bundle, i.e. a stability condition. There is now a large body of literature on how to construct a moduli space associated to a stability condition, and we build upon that literature to construct a collection of extensions of $\mathcal{J}_{g,n}(\phi)$ indexed by a linear algebra parameter ϕ.

The moduli spaces we construct are moduli spaces over the moduli space $\mathcal{M}_{g,n}^{(0)} \subset \overline{\mathcal{M}}_{g,n}$ of treelike curves instead of the moduli space of all stable curves because, for our purposes, these moduli spaces are best suited to studying the theta divisor. Extensions of the theta divisor are determined by their restriction over $\mathcal{M}_{g,n}^{(0)}$ because, quite generally, the Chow class of divisor is determined by its restriction to the complement of a closed substack of codimension at least 2. Furthermore, we prove that the different extensions of $\mathcal{J}_{g,n}$ over $\mathcal{M}_{g,n}^{(0)}$ are closely related to the different extensions of Θ, and this relationship becomes less transparent when working over $\overline{\mathcal{M}}_{g,n}$. We also assume $n > 0$ because $s_{\vec{d}}$ is otherwise undefined. These assumptions are not essential to the methods of this paper, and the authors expect the techniques of this paper can be used to e.g. construct moduli spaces over $\overline{\mathcal{M}}_{g,n}$; we discuss some of the issues in greater depth in Section 3.4.

We construct the extensions of $\mathcal{J}_{g,n}$ in Section 3. There we construct an affine space $V_{g,n}^{(0)}$, *the stability space*, and for every nondegenerate element $\phi \in V_{g,n}^{(0)}$ a family of moduli spaces $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ extending the family $\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}$ of Jacobians. The stability formalism is inspired by Oda and Seshadri [OS79].

The affine space $V_{g,n}^{(0)}$ decomposes into a *stability polytope decomposition*, a decomposition into polytopes such that $\overline{\mathcal{J}}_{g,n}(\phi_1) = \overline{\mathcal{J}}_{g,n}(\phi_2)$ if and only if ϕ_1 and ϕ_2 lie in a common polytope. For every nondegenerate $\phi \in V_{g,n}^{(0)}$, we construct a divisor $\overline{\Theta}(\phi)$ extending Θ and then we describe the dependence of the associated Chow class $\overline{\theta}(\phi)$ on ϕ as follows. For any two nondegenerate stability parameters ϕ_1 and ϕ_2, there is a distinguished isomorphism between the Chow groups $\mathrm{A}^1(\overline{\mathcal{J}}_{g,n}(\phi_1)) \cong \mathrm{A}^1(\overline{\mathcal{J}}_{g,n}(\phi_2))$, which allows to form the difference in the Chow group.

Our main result describes this difference. By Lemmas 2 and 6 in Section 3, crossing a wall in the stability space $V_{g,n}^{(0)}$ corresponds to changing the stable bidegree of a line bundle on a general element of one divisor $\Delta_{i,S} \subset \overline{\mathcal{M}}_{g,n}$ from $(d-1, g-d)$ to $(d, g-1-d)$, and leaving the stable bidegree on the general element of $\Delta_{i',S'}$ unchanged for all other $(i', S') \neq (i, S)$. Let ϕ_1 be a stability parameter in $V_{g,n}^{(0)}$ corresponding to the first stability condition, and let ϕ_2 be a stability parameter corresponding to the second one. Our main result is Theorem 17 in Section 4: the wall-crossing formula

Theorem.

$$\overline{\theta}(\phi_2) - \overline{\theta}(\phi_1) = (d-i) \cdot \delta_{i,S}. \tag{5}$$

We use this formula in Section 5 to study different extensions of $D_{\vec{d}}$. Specifically, for every nondegenerate stability parameter ϕ the morphism $s_{\vec{d}}$ extends *uniquely* to a morphism

$$s_{\vec{d}}\colon \mathcal{M}_{g,n}^{(0)} \to \overline{\mathcal{J}}_{g,n}(\phi), \tag{6}$$

so we can form the preimage

$$\overline{D}_{\vec{d}}(\phi) := s_{\vec{d}}^{-1}(\overline{\Theta}(\phi)).$$

We compute the class of these divisors:

Theorem. *For a nondegenerate stability parameter ϕ, we have*

$$[D_{\vec{d}}(\phi)] = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j + \sum_{i,S} \left(\binom{d(i,S)-i+1}{2} - \binom{d_S-i+1}{2} \right) \cdot \delta_{i,S}, \tag{7}$$

where $d(i,S)$ is the unique integer such that $(d(i,S), g-1-d(i,S))$ is the bidegree of a ϕ-stable line bundle on a general element of $\Delta_{i,S} \subset \overline{\mathcal{M}}_{g,n}$.

This is Theorem 23. We describe the relation between the divisors $[\overline{D}_{\vec{d}}(\phi)]$ and $[\overline{D}_{\vec{d}}(\mathrm{Ha})]$, $[\overline{D}_{\vec{d}}(\mathrm{SP})]$, $[\overline{D}_{\vec{d}}(\mathrm{Mü})]$ in Sections 5.1, 5.2, 5.3 respectively. In particular, we prove the following new result:

Corollary. *The pullback of the theta divisor of the family of stable semiabelic pairs extending $(\mathcal{J}_{g,n}, \Theta)$ satisfies*

$$
\begin{aligned}
[\overline{D}_{\vec{d}}(SP)] &= -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j - \sum_{i,S} \binom{d_S - i + 1}{2} \cdot \delta_{i,S} \\
&= [\overline{D}_{\vec{d}}(Ha)] - \frac{\delta_{irr}}{8} \\
&= [\overline{D}_{\vec{d}}(\phi)] \quad \textit{for any } \phi \textit{ satisfying Lemma 16.}
\end{aligned}
\tag{8}
$$

This is Corollary 25. As is explained in Section 5.2, this is also the pullback of the theta divisor studied in Caporaso's works [Cap08a, Cap09].

After this paper was first posted to the arXiv, the authors were made aware of related work of Bashar Dudin. In [Dud15], Dudin computes the pullback of the theta divisor of certain compactified universal Jacobians that are constructed by Melo in an upcoming paper. He computes the pullback of such a theta divisor to be the class in Equation (8), and the authors expect that the restriction of Melo's family to $\mathcal{M}_{g,n}^{(0)}$ is $\overline{\mathcal{J}}_{g,n}(\phi)$ for a ϕ satisfying the conditions of Lemma 16. The authors first became aware of Dudin's work on July 14, 2015. The authors first posted their preprint to the arXiv on July 13, 2015 and first publicly presented their work in a seminar on March 10, 2015. Dudin posted his paper to the arXiv on May 12, 2015.

2. Conventions

A **curve** over a field $\operatorname{Spec}(F)$ is a $\operatorname{Spec}(F)$-scheme $C/\operatorname{Spec}(F)$ that is proper over $\operatorname{Spec}(F)$, geometrically connected, and pure of dimension 1. A curve $C/\operatorname{Spec}(F)$ is a **nodal curve** if C is geometrically reduced and the completed local ring of $C \otimes \overline{F}$ at a non-regular point is isomorphic to $\overline{F}[[x,y]]/(xy)$. Here $\overline{F}$ is an algebraic closure of F.

A **family of curves** over a k-scheme T is a proper, flat morphism $C \to T$ whose fibers are curves. A family of curves $C \to T$ is a **family of nodal curves** if the fibers are nodal curves.

If F is a rank 1, torsion-free sheaf on a nodal curve C with irreducible components $\{C_i\}$, then we define the multidegree by $\deg(F) \coloneqq (\deg(F_{C_i}))$. Here F_{C_i} is the maximal torsion-free quotient of $F \otimes \mathcal{O}_{C_i}$.

Given a ring R and a set S, we write R^S for the R-module of functions $S \to R$, a free R-module with basis indexed by S.

2.1. **Graphs.** A graph Γ is a tuple $(\mathrm{Vert}, \mathrm{HalfEdge}, \mathrm{a}, \mathrm{i})$ consisting of a set of vertices Vert, a set of half-edges HalfEdge, an assignment function $\mathrm{a}\colon \mathrm{HalfEdge} \to \mathrm{Vert}$, and an involution $\mathrm{i}\colon \mathrm{HalfEdge} \to \mathrm{HalfEdge}$. The edge set is defined as the quotient set $\mathrm{Edge} \coloneqq \mathrm{HalfEdge}/\mathrm{i}$.

The endpoints of an edge $e \in \mathrm{Edge}$ are defined to be $v_1 = a(h_1)$ and $v_2 = a(h_2)$, where $\{h_1, h_2\}$ is the equivalence class represented by e. A loop based at v is an edge whose endpoints both equal v.

A n-marked graph is a graph Γ together with a (genus) map $g\colon \mathrm{Vert}(\Gamma) \to \mathbb{N}$ and a (markings) map $p\colon \{1, \dots, n\} \to \mathrm{Vert}(\Gamma)$. We call $g(v)$ the genus of $v \in \mathrm{Vert}(\Gamma)$. If $v = p(j)$, then we say that the marking j lies on the vertex v. A subgraph Γ' is always assumed to be proper ($\mathrm{Vert}(\Gamma') \subsetneq \mathrm{Vert}(\Gamma)$) and complete (for all $v' \in \mathrm{Vert}(\Gamma')$, if

$h \in \mathrm{HalfEdge}(\Gamma)$ and $a(h) = v'$, then $h \in \mathrm{HalfEdge}(\Gamma')$) and it is given the induced genus and marking maps.

We say that an n-marked graph is **stable** if it is connected, and if for all v with $g(v) = 0$, the sum of the number of edges with v as an endpoint plus the number of markings lying on v is at least 3 (when counting edges, count a loop based at v twice). The (arithmetic) **genus** of the graph is $g(\Gamma) := \sum_{v \in \mathrm{Vert}(\Gamma)} g(v) - \# \mathrm{Vert}(\Gamma) + \# \mathrm{Edge}(\Gamma) + 1$.

If Γ is a n-marked graph and $e \in \mathrm{Edge}(\Gamma)$ is an edge, the contraction of e in Γ is the graph Γ' where the half-edges corresponding to e are removed, the two endpoints w_1 and w_2 of e are replaced by a unique vertex w', and the genus and marking functions are extended to w' by $p'(j) := w'$ whenever $p(j)$ equals w_1 or w_2, and

$$g'(w') := \begin{cases} g(w_1) + g(w_2) & \text{when } e \text{ is not a loop,} \\ g(w_1) + g(w_2) + 1 & \text{when } e \text{ is a loop.} \end{cases}$$

2.2. **Moduli of curves.** Throughout the paper, we fix integers $g \geq 0$ and $n \geq 1$ (if $g = 0$, then $n \geq 3$).

Definition 1. If $(C, p_1, \ldots, p_n)$ is a stable marked curve, we define the **dual graph** Γ_C to be the n-marked graph whose vertices are the irreducible components of C decorated by their geometric genera and whose edges are the nodes of C. The loop-free dual graph $\overline{\Gamma}_C$ is the graph obtained from Γ_C by contracting all loops. We say that Γ_C (or alternatively C) is **treelike** if $\overline{\Gamma}_C$ is a tree. If $(p_1, \ldots, p_n)$ are markings of C, then we define the corresponding markings of Γ_C to be the assignment $\{1, \ldots, n\} \to \mathrm{Vert}(\Gamma_C)$ that sends j to the irreducible component containing p_j.

Given a stable marked graph Γ, we define $\mathcal{M}_{g,n}(\Gamma)$ to be the locally closed substack of $\overline{\mathcal{M}}_{g,n}$ parameterizing curves with dual graph Γ. We define $\mathcal{M}_{g,n}^{(0)} \subset \overline{\mathcal{M}}_{g,n}$ to be the open substack parameterizing treelike curves.

In this paper we will work with several divisors and their classes in $\overline{\mathcal{M}}_{g,n}$. Because every such divisor is completely determined by its restriction to $\mathcal{M}_{g,n}^{(0)}$, we will sometimes abuse the notation and denote a divisor on $\overline{\mathcal{M}}_{g,n}$ and on $\mathcal{M}_{g,n}^{(0)}$ with the same symbol.

Definition 2. For a given pair (i, S) with $i \in \{0, \ldots, g\}$ and $S \subset \{1, \ldots, n\}$ such that $1 \in S$ and

$$\begin{aligned} \#S &\leq n - 2 \text{ if } i = g, \\ \#S &\geq 2 \text{ if } i = 0; \end{aligned} \tag{9}$$

we define $\Gamma(i, S)$ to be the graph with two vertices v_1 and v_2 and one edge connecting them, and with genera $g(v_1) = i$ and $g(v_2) = g - i$, and markings

$$p(j) = \begin{cases} v_1 & \text{if } j \in S; \\ v_2 & \text{otherwise.} \end{cases}$$

The boundary divisor $\Delta_{i,S}$ is the closure of $\mathcal{M}_{g,n}(\Gamma(i, S))$ in $\overline{\mathcal{M}}_{g,n}$. The boundary divisor Δ_{irr} is the closure of the locus of irreducible, singular curves.

The restriction of the universal curve $\pi: \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ to $\Delta_{i,S}$ has two irreducible components, and we write $\mathcal{C}_{i,S}^{+}$ for the irreducible component that contains the markings S and $\mathcal{C}_{i,S}^{-}$ for the other irreducible component.

We require that (i, S) satisfy (9) so that pairs (i, S) are in bijection with the boundary divisors distinct from Δ_{irr}.

3. Stability Conditions

In this section we define families $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n}$ that extend the universal Jacobian $\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}$ and effective divisors $\overline{\Theta}(\phi) \subset \overline{\mathcal{J}}_{g,n}(\phi)$ that extend the family of theta divisors $\Theta \subset \mathcal{J}_{g,n}$. The families are indexed by stability parameters $\phi \in V^{(0)}_{g,n}$ lying in an affine space $V^{(0)}_{g,n}$. We describe the dependence of $\overline{\mathcal{J}}_{g,n}(\phi)$ on ϕ by constructing a polytope decomposition of $V^{(0)}_{g,n}$, called the stability polytope decomposition, with the property that $\overline{\mathcal{J}}_{g,n}(\phi_1) = \overline{\mathcal{J}}_{g,n}(\phi_2)$ if and only ϕ_1 and ϕ_2 lie in a common polytope.

Our construction of the $\overline{\mathcal{J}}_{g,n}(\phi)$'s is perhaps not the first construction that one might try. A natural first approach is to define $V^{(0)}_{g,n}$ to be the relative ample cone Amp inside the relative Néron–Severi space $\operatorname{Pic}(\mathcal{C}_{g,n})_{\mathbb{R}}/\pi^* \operatorname{Pic}(\mathcal{M}^{(0)}_{g,n})_{\mathbb{R}}$ and then for $\phi \in \mathrm{Amp}$ to set $\overline{\mathcal{J}}_{g,n}(\phi)$ equal to the moduli space of degree $g-1$ rank 1, torsion-free sheaves that are slope semistable with respect to ϕ. For our purposes, this does not lead to a satisfactory theory because, as was observed in [Ale04, 1.7], the condition of slope stability with respect to ϕ on degree $g-1$ sheaves is independent of ϕ, so this approach produces only one family $\overline{\mathcal{J}}_{g,n}(\phi)$. Furthermore, this family is a stack with points that have positive dimensional stabilizers (or is a highly singular coarse moduli scheme depending on how one tries to construct $\overline{\mathcal{J}}_{g,n}(\phi)$) because there are sheaves that are strictly semistable, and the presence of positive dimensional stabilizers complicates the intersection theory of $\overline{\mathcal{J}}_{g,n}(\phi)$ (see e.g. [EGS13]). Below we modify this (unsuccessful) approach to construct a large collection of families $\overline{\mathcal{J}}_{g,n}(\phi)$ that are smooth Deligne–Mumford stacks.

This section is organized as follows. In Section 3.1 we define ϕ-stability and related concepts, in Section 3.2 we define the stability polytope decomposition, and then in Section 3.3 we construct the family $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n}$ of compactified Jacobians and the family of theta divisors $\overline{\Theta}(\phi) \subset \overline{\mathcal{J}}_{g,n}(\phi)$ associated to a nondegenerate stability parameter ϕ. Finally in Section 3.4 we make some remarks about the definition of $V^{(0)}_{g,n}$, $\overline{\mathcal{J}}_{g,n}(\phi)$ and their relations to constructions from the literature.

3.1. **Stability Conditions: The Stability Space.**

The stability condition we study is the following.

Definition 3. Given a stable marked graph Γ of genus g, define $V(\Gamma) \subset \mathbb{R}^{\mathrm{Vert}(\Gamma)}$ to be the affine subspace of vectors ϕ satisfying

$$\sum_{v \in \mathrm{Vert}(\Gamma)} \phi(v) = g - 1.$$

If C is a stable marked curve with dual graph Γ and $C_0 \subset C$ is a subcurve with dual graph $\Gamma_0 \subset \Gamma$, then we write $\deg_{C_0}(F)$ or $\deg_{\Gamma_0}(F)$ for the degree of the maximal torsion-free quotient of $F \otimes \mathcal{O}_{C_0}$ and $C_0 \cap C_0^c$ or $\Gamma_0 \cap \Gamma_0^c$ for the set of edges $e \in \mathrm{Edge}(\Gamma)$ that join a vertex of Γ_0 to a vertex of its complement Γ_0^c.

Given $\phi \in V(\Gamma)$ we define a degree $g-1$ rank 1, torsion-free sheaf F on a curve C/k defined over an algebraically closed field to be **ϕ-semistable** (resp. **ϕ-stable**) if

$$\deg_{\Gamma_0}(F) \geq \sum_{v \in \mathrm{Vert}(\Gamma_0)} \phi(v) - \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2} \quad \text{(resp. } >\text{)} \tag{10}$$

for all proper subgraphs $\Gamma_0 \subset \Gamma$. We say that $\phi \in V(\Gamma)$ is **nondegenerate** if every ϕ-semistable sheaf is ϕ-stable.

Remark 1. Nondegenerate ϕ's exist since e.g. any ϕ with irrational coefficients must be general.

Remark 2. We have defined a sheaf F to be ϕ-semistable if an explicit lower bound on $\deg_{\Gamma_0}(F)$ holds, but this condition is equivalent to an explicit upper bound. Given a subcurve $C_0 \subset C$ with dual graph $\Gamma_0 := \Gamma(C_0)$ and a rank 1, torsion-free sheaf F of degree $g-1$, we have $\deg_{C_0}(F) + \deg_{C_0^c}(F) = g - 1 - \delta_{\Gamma_0}(F)$ for $\delta_{\Gamma_0}(F)$ the number of nodes $p \in \Gamma_0 \cap \Gamma_0^c$ such that the stalk of F at p fails to be locally free. As a consequence, the ϕ-semistability (resp. ϕ-stability) inequality can be rewritten as

$$\deg_{\Gamma_0}(F) \leq \sum_{v \in \mathrm{Vert}(\Gamma_0)} \phi(v_0) - \delta_{\Gamma_0}(F) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2} \quad \text{(resp. } <\text{).} \tag{11}$$

If we combine Inequalities (10) and (11), then we get a third formulation of ϕ-semistability (resp. ϕ-stability): F is ϕ-semistable (resp. ϕ-stable) if and only if

$$\left| \deg_{\Gamma_0}(F) - \sum_{v \in \mathrm{Vert}(\Gamma_0)} \phi(v_0) + \frac{\delta_{\Gamma_0}(F)}{2} \right| \leq \frac{\#(\Gamma_0 \cap \Gamma_0^c) - \delta_{\Gamma_0}(F)}{2} \quad \text{(resp. } <\text{).} \tag{12}$$

The following lemma shows that slope semistability is a special case of ϕ-semistability.

Lemma 1. *Let $(C, p_1, \ldots, p_n)$ be a stable marked curve and A and M line bundles on C with A ample. If $\phi(A, M) \in V(\Gamma_C)$ is defined by setting for $v \in \mathrm{Vert}(\Gamma_C)$*

$$\phi(A, M)(v) := \frac{\deg_v(A)}{\deg(A)} \deg(M) + \frac{\deg_v(\omega_X)}{2} - \deg_v(M), \tag{13}$$

then a degree $g-1$ rank 1, torsion-free sheaf F is $\phi(A, M)$-semistable (resp. $\phi(A, M)$-stable) if and only if $F \otimes M$ is slope semistable (resp. stable) with respect to A.

Proof. By elementary algebra, this is a consequence of the explicit computation of semistability in [Ale04, pages 1245–1246] or [CMKV12]. □

Motivated by the lemma, we make the following definition.

Definition 4. We define the **canonical parameter** $\phi_{\mathrm{can}} \in V(\Gamma)$ of a stable marked curve C with dual graph Γ by setting $\phi_{\mathrm{can}}(v) = \frac{\deg_v \omega_C}{2}$ for $v \in \mathrm{Vert}(\Gamma)$.

Concretely $\phi_{\mathrm{can}}(v) = g(v) - 1 + \#\,\mathrm{Edge}(N_v)/2$ where $N_v \subset \Gamma$ is the neighbourhood of v. A sheaf F is ϕ_{can}-semistable if and only if it is slope semistable with respect to an ample line bundle, i.e. $\phi_{\mathrm{can}} = \phi(A, \mathcal{O}_C)$ for some (equivalently all) ample A.

Next we define a stability space that controls families $\overline{\mathcal{J}}_{g,n}(\phi)$ over $\mathcal{M}_{g,n}^{(0)}$.

Definition 5. Suppose that $c: \Gamma_1 \to \Gamma_2$ is a contraction of stable marked graphs. We say that $\phi_1 \in V(\Gamma_1)$ is **compatible** with $\phi_2 \in V(\Gamma_2)$ with respect to c if

$$\phi_2(v_2) = \sum_{c(v_1)=v_2} \phi_1(v_1) \tag{14}$$

for all vertices $v_2 \in \mathrm{Vert}(\Gamma_2)$.

Define the **stability space** to be the subset of

$$V^{(0)}_{g,n} \subset \prod_{b_1(\overline{\Gamma})=0} V(\Gamma) \tag{15}$$

that consists of vectors $\phi = (\phi(\Gamma))$ such that $\phi(\Gamma_1)$ is compatible with $\phi(\Gamma_2)$ with respect to every contraction $c: \Gamma_1 \to \Gamma_2$.

The **canonical parameter** $\phi_{\mathrm{can}} \in V^{(0)}_{g,n}$ is defined to be $\phi_{\mathrm{can}} := (\phi_{\mathrm{can}}(\Gamma))$.

Given $\phi \in V^{(0)}_{g,n}$ we say that a degree $g-1$ rank 1, torsion-free sheaf F on a stable marked curve $(C, p_1, \dots, p_n) \in \mathcal{M}^{(0)}_{g,n}$ is **ϕ-semistable** (resp. **ϕ-stable**) if F is $\phi(\Gamma)$-semistable (resp. $\phi(\Gamma)$-stable) for Γ the dual graph of C. We say that $\phi \in V^{(0)}_{g,n}$ is **nondegenerate** if $\phi(\Gamma)$ is nondegenerate for all Γ.

Remark 3. In addition to compatibility with contractions, a natural condition to impose on a stability parameter is that it is invariant under automorphisms, i.e. $\phi(\Gamma)(v) = \phi(\Gamma)(\alpha(v))$ for all graphs Γ and all graph automorphisms $\alpha: \Gamma \to \Gamma$. We believe this condition follows from compatibility with contractions, although this becomes false if n is allowed to be 0. We do not pursue these issues here because they are not needed.

Remark 4. In Definition 5 we defined $V^{(0)}_{g,n}$ to be the subset of ϕ's that are compatible with contractions in order to ensure that there is a well-behaved moduli stack $\overline{\mathcal{J}}_{g,n}(\phi)$ associated to a nondegenerate stability parameter ϕ (the existence of $\overline{\mathcal{J}}_{g,n}(\phi)$ is Proposition 11 below). Without the compatibility condition, a suitable moduli stack may not exist. The essential point is this: Suppose F_η is a line bundle on a stable marked curve C_η that specializes to F_s on C_s within some 1-parameter family. If $C_{0,\eta}$ is an irreducible component of C_η, then that irreducible component specializes to a subcurve $C_{0,s}$, and the degrees are related by

$$\deg_{C_{0,\eta}}(F_\eta) = \deg_{C_{0,s}}(F_s) \tag{16}$$

(by continuity of the Euler characteristic).

Equation (16) is exactly the condition that the degree vector $\deg(F)$ is compatible with the contraction $c: \Gamma_{C_s} \to \Gamma_{C_\eta}$. Thus when defining stability conditions on line bundles, it is natural to require that the degree vectors of stable line bundles are compatible with contractions, and this holds when the line bundles are the ϕ-stable line bundles for a stability parameter ϕ that is compatible with contractions.

We conclude the section by proving that a nondegenerate stability parameter $\phi \in V^{(0)}_{g,n}$ is determined by its components $\phi(\Gamma)$ for Γ a 2-vertex graph.

Lemma 2. *The restriction of the natural projection*

$$\prod_{b_1(\overline{\Gamma})=0} V(\Gamma) \to \prod_{\substack{\#\,\mathrm{Vert}(\Gamma)=2 \\ b_1(\overline{\Gamma})=0}} V(\Gamma) \tag{17}$$

to $V_{g,n}^{(0)}$ is a bijection.

Proof. To begin, we examine the condition that

$$\phi \in \prod_{b_1(\overline{\Gamma})=0} V(\Gamma)$$

is compatible with contractions. Let Γ be a stable graph with $b_1(\overline{\Gamma}) = 0$. If $e \in \mathrm{Edge}(\Gamma)$ is an edge that is not a loop, then $\Gamma - e$ has two connected components, say Γ^+ and Γ^-. If c is the contraction that contracts all edges of Γ except for e, then Γ^+ and Γ^- are contracted to two distinct vertices, say v^+ and v^- respectively. The vector ϕ is compatible with c if and only if the following equalities are satisfied:

$$\begin{aligned} \sum_{v \in \mathrm{Vert}(\Gamma^+)} \phi(\Gamma)(v) - \sum_{v \in \mathrm{Vert}(\Gamma^-)} \phi(\Gamma)(v) &= \phi(c(\Gamma))(v^+) - \phi(c(\Gamma))(v^-), \\ \sum_{v \in \mathrm{Vert}(\Gamma)} \phi(\Gamma)(v) &= g - 1. \end{aligned} \tag{18}$$

Varying over all nonloops $e \in \mathrm{Edge}(\Gamma)$, the equations in (18) form a system of $\#\,\mathrm{Edge}(\overline{\Gamma})+1 = \#\,\mathrm{Vert}(\Gamma)$ inhomogeneous equations in $\#\,\mathrm{Vert}(\Gamma)$ variables. Furthermore, the associated system of homogeneous equations is nondegenerate (induct on $\#\,\mathrm{Edge}(\Gamma)$ to shows that if $v_0 \in \mathrm{Vert}(\Gamma)$ is a leaf, then the determinant of the system associated to Γ equals twice the determinant associated to the graph obtained by contracting the unique non-loop containing v_0). In particular, $\phi(\Gamma) \in V(\Gamma)$ is the unique vector satisfying (18). It immediately follows that the projection (17) is injective.

We establish surjectivity as follows. Given

$$\phi \in \prod_{\substack{\#\,\mathrm{Vert}(\Gamma)=2 \\ b_1(\overline{\Gamma})=0}} V(\Gamma),$$

define, for Γ a stable marked graph with $b_1(\overline{\Gamma}) = 0$, $\phi(\Gamma)$ to be the unique solution to (18) and then define

$$\phi := (\phi(\Gamma)) \in \prod_{b_1(\overline{\Gamma})=0} V(\Gamma).$$

This vector is compatible with all contractions. Indeed, given a contraction $c\colon \Gamma_1 \to \Gamma_2$, define $\phi'(\Gamma_2) \in V(\Gamma_2)$ by setting

$$\phi'(\Gamma_2)(v_2) = \sum_{c(v_1)=v_2} \phi(\Gamma_1)(v_2).$$

Then both $\phi'(\Gamma_2)$ and $\phi(\Gamma_2)$ satisfy (18), so $\phi'(\Gamma) = \phi(\Gamma)$, proving that

$$\phi := (\phi(\Gamma)) \in \prod_{b_1(\overline{\Gamma})=0} V(\Gamma)$$

is compatible with contractions and thus the surjectivity of (17). $\square$

3.2. **Stability Conditions: The stability polytope decomposition.** Here we define the polytope decompositions of $V(\Gamma)$ and $V_{g,n}^{(0)}$ that describe how ϕ-stability depends on ϕ.

We will use the following definition and lemma to construct the decomposition.

Definition 6. A subgraph $\Gamma_0 \subset \Gamma$ is said to be **elementary** if both Γ_0 and its complement Γ_0^c are connected.

Remark 5. The vertex set of an elementary subgraph is an elementary cut in the sense of [OS79, page 31].

Remark 6. When $b_1(\overline{\Gamma}) = 0$ (the case of present interest), a subgraph $\Gamma_0 \subset \Gamma$ is elementary if and only if $\Gamma_0 \cap \Gamma_0^c$ consists of a single edge.

Lemma 3. *Let $(C, p_1, \ldots, p_n)$ be a stable marked curve and $\phi \in V(\Gamma_C)$. A rank 1, torsion-free sheaf F is ϕ-semistable (resp. ϕ-stable) if and only if Inequality (10) holds for all elementary subgraphs Γ_0 of Γ_C.*

Proof. Set $\Gamma := \Gamma_C$. It is enough to show that if F satisfies Inequality (12) for all elementary subgraphs $\Gamma_0 \subset \Gamma$, then it satisfies the inequality for *all* subgraphs. First, consider the case where Γ_0^c is connected. Let $\Gamma_1, \ldots, \Gamma_n$ be the connected components of Γ_0.

Each of the connected components $\Gamma_1, \ldots, \Gamma_n$ is an elementary subgraph of Γ. Indeed, the complement of Γ_i in Γ is

$$\Gamma_i^c = \Gamma_0^c \cup \Gamma_1 \cup \Gamma_2 \cdots \cup \Gamma_{i-1} \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_n,$$

and we can connect each Γ_j for $j \neq i$ to Γ_0^c as follows. Since Γ is connected, for $j \neq i$, we can connect any vertex in Γ_j to any vertex in Γ_0^c by a path in Γ. Pick one such path $v_0, v_1, \ldots, v_n$ that has minimal length. There is no consecutive pair v_i, v_{i+1} of vertices with $v_i \in \text{Vert}(\Gamma_{k_1})$ and $v_{i+1} \in \text{Vert}(\Gamma_{k_2})$ for distinct k_1, k_2 because no edge joins Γ_{k_1} to Γ_{k_2}. Furthermore, the first vertex that lies in Γ_0^c must be v_n by minimality, so the vertices $v_1, \ldots, v_{n-1}$ must all lie in Γ_j. This proves that Γ_i is elementary.

By hypothesis, Inequality (12) holds for the subgraphs $\Gamma_1, \ldots, \Gamma_n$ and combining those inequalities with the triangle inequality, we get Inequality (12) for Γ_0. This proves the lemma under the assumption that Γ_0^c is connected.

For arbitrary Γ we argue as follows. Inequality (12) is symmetric with respect to replacing Γ_0 with Γ_0^c, so the result follows immediately when Γ_0 is connected and from the case that Γ_0 is connected, the general case then follows by expressing Γ_0 as a union of connected components and applying the triangle inequality. □

Definition 7. Let Γ be a stable marked graph of genus g. To a subgraph $\Gamma_0 \subset \Gamma$ and an integer $d \in \mathbb{Z}$ we associate the affine linear function $\ell(\Gamma_0, d) \colon V(\Gamma) \to \mathbb{R}$ defined by

$$\ell(\Gamma_0, d)(\phi) := d - \sum_{v \in \Gamma_0} \phi(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2}.$$

We call

$$H(\Gamma_0, d) := \{\phi \in V(\Gamma) \colon \ell(\Gamma_0, d)(\phi) = 0\}$$

g_1 g_2

FIGURE 1. A tree Γ with two vertices of genera g_1 and g_2, with $g_1 + g_2 = g$

a **stability hyperplane** if $\Gamma_0 \subset \Gamma$ is an elementary subgraph. A connected component of the complement of all stability hyperplanes in $V(\Gamma)$

$$V(\Gamma) - \bigcup_{\substack{\Gamma_0 \subset \Gamma \text{ elementary} \\ d \in \mathbb{Z}}} \{\phi \in V(\Gamma) \colon \ell(\Gamma_0, d)(\phi) = 0\}$$

is defined to be a **stability polytope**, and the set of all stability polytopes is defined to be the **stability polytope decomposition** of $V(\Gamma)$.

By definition if ϕ_0 is a nondegenerate stability parameter, the stability polytope $\mathcal{P}$ containing ϕ_0 can be written as:

$$\mathcal{P} = \{\phi \in V(\Gamma) \colon \ell(\Gamma_0, d)(\phi) > 0 \text{ for all } \ell(\Gamma_0, d) \text{ s.t. } \ell(\Gamma_0, d)(\phi_0) > 0\}. \tag{19}$$

The stability polytope $\mathcal{P}$ is a rational bounded convex polytope because in Equation (19) only finitely many $\ell(\Gamma_0, d)$'s are needed to define $\mathcal{P}$.

Example 1. When Γ_C has a single vertex, $V(\Gamma)$ is a 0-dimensional affine space. There are no elementary subgraphs of Γ, so there is only one stability polytope: $V(\Gamma)$ itself.

Example 2. Suppose that Γ is the graph depicted in Figure 1. The associated stability polytopes are depicted in Figure 2. The affine space $V(\Gamma)$ is 1-dimensional, and a stability polytope is an open line segment with endpoints at two consecutive half-integer points. More precisely, if $\vec{d} = (d(v_1), d(v_2)) \in V_{\mathbb{Z}}(\Gamma)$ is an integral vector, then the set of solutions to

$$\begin{aligned} d(v_1) - \phi(v_1) + 1/2 &> 0, \\ d(v_2) - \phi(v_2) + 1/2 &> 0 \end{aligned}$$

is a stability polytope $\mathcal{P}(d)$ that can be described as the relative interior of the convex hull of $(d(v_1) - 1/2, d(v_2) + 1/2)$ and $(d(v_1) + 1/2, d(v_2) - 1/2)$, and every stability polytope can be written as $\mathcal{P}(\vec{d})$ for a unique $\vec{d}$.

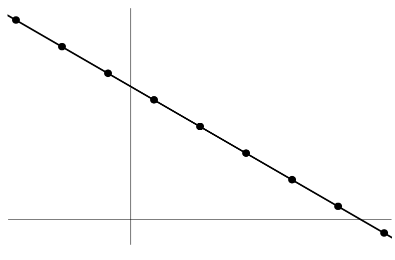

FIGURE 2. The stability polytopes of a two-vertex graph Γ

One property of the graph depicted in Figure 1 is that, for every nondegenerate stability parameter ϕ, there is a unique ϕ-stable multidegree. This is more generally true for treelike graphs:

Lemma 4. *Let Γ be a treelike graph and $\phi \in V(\Gamma)$ a nondegenerate stability parameter. Then there exists a ϕ-stable line bundle F. Furthermore, any two ϕ-stable line bundles have the same multidegree.*

Proof. We first prove that any two ϕ-stable line bundles have the same multidegree. Endow $\overline{\Gamma}$ with the structure of a rooted tree by arbitrarily picking a vertex $r_0 \in \operatorname{Vert}(\overline{\Gamma})$ as the root. We prove the lemma by working one vertex at a time, starting with the leaves and ending with the root. Suppose $v_0 \in \operatorname{Vert}(\overline{\Gamma})$ is a vertex that is not the root r_0. Define $\Gamma_0 \subset \Gamma$ to be the induced subgraph on the set of vertices consisting of v_0 and its descendants. For a line bundle F, the ϕ-stability inequality (12) for Γ_0 takes the form

$$|\deg_{\Gamma_0}(F) - \sum_{v \in \Gamma_0} \phi(v)| < \frac{1}{2}. \tag{20}$$

This proves that the partial degree $\deg_{\Gamma_0}(F)$ of a ϕ-stable line bundle is uniquely determined (and equal to the integer nearest to $\sum_{v \in \Gamma_0} \phi(v)$). Since $\deg_{v_0}(F) = \deg_{\Gamma_0}(F) - \deg_{\Gamma_1}(F)$ for $\Gamma_1 := \Gamma_0 - \{v_0\}$, it follows by reverse induction on the depth of v_0 that $\deg_{v_0}(F)$ is also uniquely determined. Given that $\deg_{v_0}(F)$ is uniquely determined for all $v_0 \neq r_0$, we can conclude that $\deg_{r_0}(F)$ is also uniquely determined as $\deg_{r_0}(F) = g - 1 - \deg_{\Gamma_2}(F)$ for $\Gamma_2 := \Gamma - \{r_0\}$. This proves any two ϕ-stable line bundles have the same multidegree. For existence, observe that we can inductively construct a ϕ-semistable line bundle F by requiring that $\deg_{\Gamma_0}(F)$ equals the nearest integer to $\sum_{v \in \Gamma_0} \phi(v)$ (i.e. equals the unique solution to Inequality (20)). □

Lemma 5. *Let $(C, p_1, \ldots, p_n)$ be a stable marked curve such that the dual graph Γ_C is treelike. Given two nondegenerate stability parameters $\phi_1, \phi_2, \in V(\Gamma_C)$, ϕ_1-stability coincides with ϕ_2-stability if and only if there exists a stability polytope containing both ϕ_1 and ϕ_2.*

Proof. If ϕ_1, ϕ_2 both lie in a stability polytope, then ϕ_1-stability coincides with ϕ_2-stability by Lemma 3. Conversely suppose that ϕ_1 and ϕ_2 lie in distinct stability polytopes. By definition there exists $d \in \mathbb{Z}$ and $\Gamma_0 \subset \Gamma$ an elementary subgraph such that $\ell(\Gamma_0, d)(\phi_1) > 0$ but $\ell(\Gamma_0, d)(\phi_2) < 0$. There exists (by Lemma 4) a ϕ_1-stable line bundle F, and its multidegree satisfies

$$|\deg_{\Gamma_0}(F) - \sum_{v \in \Gamma_0} \phi_1(v_0)| < 1/2,$$

i.e. $\deg_{\Gamma_0}(F)$ is the least integer greater than $\sum\limits_{v \in \Gamma_0} \phi_1(v_0) - 1/2$. In particular, $\deg_{\Gamma_0}(F) \leq d$, so

$$\begin{aligned}
\deg_{\Gamma_0}(F) - \sum_{v \in \Gamma_0} \phi_2(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2} &= (\deg_{\Gamma_0}(F) - d) + (d - \sum_{v \in \Gamma_0} \phi_2(v) + \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2}) \\
&= (\deg_{\Gamma_0}(F) - d) + \ell(\Gamma_0, d)(\phi_2) \\
&< 0,
\end{aligned}$$

and F is not ϕ_2-stable. □

Remark 7. Lemma 5 becomes false if the stability hyperplanes are defined to be the subsets $\{\ell(\Gamma_0, d)(\phi) = 0\}$ with $\Gamma_0 \subset \Gamma$ a possibly non-elementary subgraph. For example, if Γ is the graph depicted in Figure 3, then the decomposition by black rectangles depicted in Figure 4 is the stability polytope decomposition of $V(\Gamma)$ (or more precisely its isomorphic image under the projection $V(\Gamma) \to \mathbb{R}^2$, $\phi \mapsto (\phi(v_1), \phi(v_2))$). The subdivision of the polytope decomposition given by the dotted and solid lines is the decomposition by the hyperplanes $\{\ell(\Gamma_0, d)(\phi) = 0\}$ with $\Gamma_0 \subset \Gamma$ a possibly non-elementary subgraph.

Suppose that $\Gamma = \Gamma_C$ is the dual graph of C. When $\phi \in V(\Gamma)$ crosses a dotted line, the set of vectors $\vec{d} \in \mathbb{Z}^{\mathrm{Vert}(\Gamma)}$ satisfying $|\vec{d}(\Gamma_0) - \sum_{v \in \Gamma_0} \phi(v_0)| \leq \frac{\#(\Gamma_0 \cap \Gamma_0^c)}{2}$ changes, but the subset of vectors of the form $\vec{d} = \deg(F)$ for F a ϕ-semistable sheaf does not change.

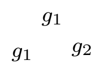

FIGURE 3. A tree Γ with three vertices, $g_1 + g_2 + g_3 = g$

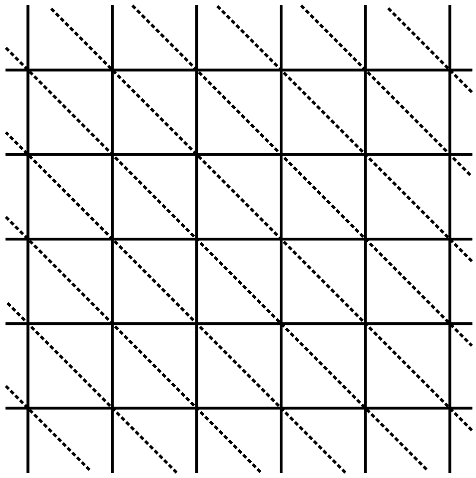

FIGURE 4. The stability polytopes of $V(\Gamma)$

We now define the combinatorial objects that control the stability conditions over $\mathcal{M}_{g,n}^{(0)}$.

Definition 8. For Γ a stable marked graph of genus g with $b_1(\overline{\Gamma}) = 0$, $\Gamma_0 \subset \Gamma$ an elementary subgraph, and $d \in \mathbb{Z}$ an integer, we define

$$\{\phi \in V_{g,n}^{(0)} : \ell(\Gamma_0, d)(\phi(\Gamma)) = 0\}$$

to be a **stability hyperplane**. A **stability polytope** for $V_{g,n}^{(0)}$ is defined to be a connected component of

$$V_{g,n}^{(0)} - \bigcup_{\substack{\Gamma_0 \subset \Gamma \text{ elementary} \\ d \in \mathbb{Z}}} \{\phi \in V_{g,n}^{(0)} : \ell(\Gamma_0, d)(\phi(\Gamma)) = 0\}.$$

The collection of all stability polytopes is defined to be the **stability polytope decomposition** of $V_{g,n}^{(0)}$.

As before, the stability polytopes in $V_{g,n}^{(0)}$ are rational bounded convex polytopes.

Having shown in Lemma 2 that a stability parameter $\phi \in V_{g,n}^{(0)}$ is determined by 2-vertex graphs, we now prove analogous statements about stability hyperplanes and polytopes.

Lemma 6. *If $H \subset V_{g,n}^{(0)}$ is a stability hyperplane, then*

$$H = H(\Gamma, d) =: H(i, S, d)$$

for $\Gamma = \Gamma(i, S)$ a stable marked graph with two vertices and one edge (see Definition 2).

Proof. Let

$$H = \{\ell(\Gamma_1, d)(\phi(\Gamma_2)) = 0\}$$

be a given stability hyperplane (so Γ_2 is a treelike graph, $\Gamma_1 \subset \Gamma_2$ an elementary subgraph, and $d \in \mathbb{Z}$ an integer).

Define $c \colon \Gamma_2 \to \Gamma$ to be the contraction that contracts Γ_1 to a vertex w_1 and Γ_0^c to a vertex w_2, so Γ is a stable marked graph with two vertices. By compatibility we have $\phi(\Gamma)(w_1) = \sum_{v \in \Gamma_1} \phi(\Gamma_2)(v)$, so

$$\begin{aligned} \{\ell(w_1, d)(\phi(\Gamma)) = 0\} =& \{\ell(\Gamma_1, d)(\phi(\Gamma_2)) = 0\} \\ =& H. \end{aligned}$$

□

Lemma 6 implies that when $\phi \in V_{g,n}^{(0)}$ varies in such a way that ϕ-semistability changes, that variation is already witnessed over a graph with two vertices. As a corollary, we obtain the following description of stability polytopes:

Corollary 7. *Given a stability polytope $\mathcal{P}(\Gamma)$ for every stable marked graph Γ with $b_1(\overline{\Gamma}) = 0$ and $\#\operatorname{Vert}(\Gamma) = 2$, there exists a unique stability polytope $\mathcal{P} \subset V_{g,n}^{(0)}$ such that the projection onto $V(\Gamma)$ is $\mathcal{P}(\Gamma)$ for all 2-vertex graphs Γ.*

Proof. This follows from Lemma 2, together with Lemma 6. Uniqueness is Lemma 2. To prove existence, by the same lemma there exists $\phi_0 \in V_{g,n}^{(0)}$ satisfying $\phi_0(\Gamma) \in \mathcal{P}(\Gamma)$ for all 2-vertex graphs Γ. By Lemma 6 ϕ_0 is not contained in a stability hyperplane, so it is contained in a unique stability polytope that satisfies the desired condition by the same lemma. □

To study the set of stability polytopes, we introduce the following group action.

Definition 9. Define $W_{\mathbb{Z}}(\Gamma) \subset \mathbb{Z}^{\operatorname{Vert}(\Gamma)}$ to be the subgroup of sum-zero vectors. The **natural action** of $W_{\mathbb{Z}}(\Gamma)$ on $V(\Gamma)$ is the translation action, $(\psi + \phi)(v) = \psi(v) + \phi(v)$.

Lemma 8. *The natural action of $W_{\mathbb{Z}}(\Gamma)$ on $V(\Gamma)$ maps stability polytopes to stability polytopes.*

Proof. This follows from the identity

$$\ell(\Gamma_0, d)(\psi + \phi) = \ell(\Gamma_0, d - \psi(\Gamma_0))(\phi).$$

□

Definition 10. We define $(W_{g,n})_{\mathbb{Z}}$ to be the additive subgroup

$$(W_{g,n})_{\mathbb{Z}} \subset \prod_{b_1(\overline{\Gamma})=0} W_{\mathbb{Z}}(\Gamma)$$

that consists of vectors satisfying the contraction compatibility condition:

$$\psi(\Gamma_2)(v_2) = \sum_{c(v_1)=v_2} \psi(\Gamma_1)(v_1)$$

for all contractions $c\colon \Gamma_1 \to \Gamma_2$ and all vertices $v_2 \in \mathrm{Vert}(\Gamma_2)$. The **natural action** of $(W_{g,n})_{\mathbb{Z}}$ on $V_{g,n}^{(0)}$ is defined by $(\psi + \phi)(\Gamma) = \psi(\Gamma) + \phi(\Gamma)$.

Lemma 9. *The natural action of $(W_{g,n})_{\mathbb{Z}}$ on $V_{g,n}^{(0)}$ maps stability polytopes to stability polytopes.*

Proof. This follows from Lemma 8. □

We define $\mathrm{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ to be the subgroup of the group $\mathrm{Pic}(\mathcal{C}_{g,n})/\pi^*(\mathrm{Pic}(\mathcal{M}_{g,n}^{(0)}))$ generated by the images of line bundles F with the property that the restriction to any fiber of $\pi\colon \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}^{(0)}$ has degree 0. The multidegree defines a homomorphism $\deg\colon \mathrm{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)}) \to (W_{g,n})_{\mathbb{Z}}$. The **natural action** of $\mathrm{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ on $V_{g,n}^{(0)}$ is the action induced by the action of $(W_{g,n})_{\mathbb{Z}}$ via the homomorphism deg. The subschemes $\mathcal{C}_{i,S}^{\pm} \subset \mathcal{C}_{g,n}$ associated with each component over $\Delta_{i,S}$ in the universal curve (see Definition 2) are effective Cartier divisors, so their associated line bundles $\mathcal{O}(\mathcal{C}_{i,S}^{\pm})$ are defined, and we use them to prove the following lemma.

Lemma 10. *The subgroup of* $\mathrm{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ *generated by the line bundles* $\mathcal{O}(\mathcal{C}_{i,S}^{\pm})$ *acts freely and transitively on the set of stability polytopes in $V_{g,n}^{(0)}$.*

Proof. Consider the line bundle $F := \mathcal{O}(\mathcal{C}_{i,S}^{+})$ associated to a pair (i, S) as in Definition 2. Its degree vector $\deg(F) \in (W_{g,n})_{\mathbb{Z}}$ satisfies

$$\deg(F)(\Gamma) = \begin{cases} (-1, +1) & \text{if } \Gamma = \Gamma(i, S); \\ (0, 0) & \text{if } \Gamma \neq \Gamma(i, S),\ \#\mathrm{Vert}(\Gamma) = 2. \end{cases} \tag{21}$$

Using the description of stability polytopes associated to a graph with two vertices in Example 2, we conclude that the subgroup generated by the $\mathcal{O}(\mathcal{C}_{i,S}^{+})$'s acts transitively on the image of $V_{g,n}^{(0)}$ in

$$\prod_{\substack{\#\mathrm{Vert}(\Gamma)=2 \\ b_1(\overline{\Gamma})=0}} V(\Gamma),$$

and from Corollary 7 we deduce that the same is true for $V_{g,n}^{(0)}$. To see that the action is free, observe that an element of $\mathrm{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ acts as translation by its multidegree, so an element acting trivially must have multidegree 0 and the only such element is the identity. □

3.3. **Stability Conditions: Representability.** We now restrict our attention to a stability parameter $\phi \in V_{g,n}^{(0)}$ that is nondegenerate. Given such a ϕ, we construct a family $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ of compactified Jacobians and a family of theta divisors $\overline{\Theta}(\phi) \subset \overline{\mathcal{J}}_{g,n}(\phi)$, and then we describe some properties of $\overline{\Theta}(\phi)$.

Definition 11. Given T a k-scheme and $(C/T, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}^{(0)}(T)$ a family of stable marked curves, a **family of degree $g-1$ rank 1, torsion-free sheaves** on C/T is a locally finitely presented $\mathcal{O}_C$-module F that is $\mathcal{O}_T$-flat and has rank 1, torsion-free fibers. Given $\phi \in V_{g,n}^{(0)}$, we say F is a **family of ϕ-semistable sheaves** if the fibers are ϕ-semistable.

Definition 12. Given $\phi \in V_{g,n}^{(0)}$, define $\overline{\mathcal{J}}_{g,n}^{\text{pre}}(\phi)$ to be the category fibered in groupoids whose objects are tuples $(C, p_1, \ldots, p_n; F)$ consisting of a family of treelike genus g, n-marked curves $(C/T, p_1, \ldots, p_n)$ and a family of ϕ-semistable rank 1, torsion-free sheaves F on C/T. The morphisms of $\overline{\mathcal{J}}_{g,n}^{\text{pre}}(\phi)$ are defined by defining a morphism from $(C, p_1, \ldots, p_n; F) \in \overline{\mathcal{J}}_{g,n}(\phi)(T)$ to $(C', p_1', \ldots, p_n'; F') \in \overline{\mathcal{J}}_{g,n}(\phi)(T')$ lying over a k-morphism $t: T \to T'$ to be a tuple consisting of an isomorphism of marked curves $\widetilde{t}: (C, p_1, \ldots, p_n) \cong (C'_T, (p_1')_T, \ldots, (p_n')_T)$, and an isomorphism of $\mathcal{O}_C$-modules $F \cong \widetilde{t}^*(F'_T)$.

With this definition, for every object $(C, p_1, \ldots, p_n; F)$ of $\overline{\mathcal{J}}_{g,n}^{\text{pre}}(\phi)(T)$ the rule that sends $g \in \mathbb{G}_m(T)$ to the automorphism of F defined by multiplication by g defines an embedding $\mathbb{G}_m(T) \to \text{Aut}(C, p_1, \ldots, p_n; F)$ that is compatible with pullbacks. The image of this embedding is contained in the center of the automorphism group, so the rigidification stack in the sense of [ACV03, Section 5.1.5] is defined, and we call this stack the **universal family of ϕ-compactified Jacobians** $\overline{\mathcal{J}}_{g,n}(\phi)$.

Proposition 11. *Assume $\phi \in V_{g,n}^{(0)}$ is nondegenerate. Then the forgetful morphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is representable, proper, and flat. In particular, $\overline{\mathcal{J}}_{g,n}(\phi)$ is a separated Deligne–Mumford stack. Furthermore, $\overline{\mathcal{J}}_{g,n}(\phi)$ is k-smooth.*

Proof. To begin, we prove the statement for one specific ϕ. Pick an odd number $B > 2g-2+n-1$ and then set,

$$\begin{aligned} b :=& B - (2g-2+n-1), \\ A :=& \omega(bp_1 + p_2 + \cdots + p_n), \text{ and} \\ M :=& \mathcal{O}(p_1). \end{aligned}$$

The authors claim that $\phi_0 := \phi(A, M)$ is nondegenerate. If the claim failed, then the expression in Equation (13) would be an integer for some proper subcurve $Y \subset X$ of a stable marked curve or equivalently the slope $\deg_{\Gamma_0}(A)/\deg(A)$ would be a half-integer for some proper subgraph $\Gamma_0 \subset \Gamma_C$, but this is impossible because the slope is of the form k/B for $k \in \mathbb{Z}$, $0 < k < B$. This proves the claim.

Lemma 1 identifies ϕ_0-stability with slope stability, so the representability result [Sim94, Theorem 1.21] implies that $\overline{\mathcal{J}}_{g,n}(\phi_0) \to \mathcal{M}_{g,n}^{(0)}$ is representable and proper (the conclusion in loc. cit. that étale locally a universal family of sheaves exists is equivalent to the representability of $\overline{\mathcal{J}}_{g,n}(\phi_0)$). The flatness of $\overline{\mathcal{J}}_{g,n}(\phi_0) \to \mathcal{M}_{g,n}^{(0)}$ and the

k-smoothness of $\overline{\mathcal{J}}_{g,n}(\phi_0)$ follow from a modification of the deformation theory argument in [CMKV12].

For an arbitrary nondegenerate ϕ, we argue as follows. By Lemma 10, for a given nondegenerate $\phi \in V_{g,n}^{(0)}$, there is a line bundle $L \in \mathrm{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ so that ϕ and $\deg(L)+\phi_0$ lie in the same stability polytope and then the rule $F \mapsto F \otimes L$ identifies $\overline{\mathcal{J}}_{g,n}(\phi_0)$ with $\overline{\mathcal{J}}_{g,n}(\phi)$. □

Remark 8. The proof of Proposition 11 shows in fact that for all nondegenerate ϕ, there exists a separated Deligne-Mumford stack $\overline{\mathcal{J}}_{g,n}^{\mathrm{comp}}(\phi)$ such that the forgetful morphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ extends to a forgetful morphism $\overline{\mathcal{J}}_{g,n}^{\mathrm{comp}}(\phi) \to \overline{\mathcal{M}}_{g,n}$, which is also representable, proper and flat. Its fiber over a geometric point $[C] \in \overline{\mathcal{M}}_{g,n}$ parameterizes ϕ_C-semistable rank 1 torsion-free sheaves on C for some stability parameter ϕ_C (as in Definition 3) that depends on ϕ and on $[C]$. In the rest of this paper, we will never need to work with families of stable curves that are not treelike.

Remark 9. When ϕ is degenerate, the authors expect that $\overline{\mathcal{J}}_{g,n}(\phi)$ is still an algebraic stack, but then the forgetful morphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is not representable, and $\overline{\mathcal{J}}_{g,n}(\phi)$ is not Deligne–Mumford. We do not pursue this issue here because we have no use for these more general families in this paper.

Proposition 12. *For $\phi \in V_{g,n}^{(0)}$ nondegenerate, the fibers of the forgetful morphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ are irreducible.*

Proof. This follows from Lemma 4. In a fiber of $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$, the locus of line bundles of fixed multidegree is k-smooth and connected (as it is a torsor for the generalized Jacobian, a semiabelian variety). Since there is a unique ϕ-stable multidegree of a line bundle, we conclude that the line bundle locus in a fiber is connected. But the line bundle locus is also dense in its fiber (since e.g. a tangent space computation shows that this locus is the k-smooth locus), so we conclude that the fiber is irreducible. □

We now turn our attention to the theta divisor and its associated Chow class. The theta divisor $\overline{\Theta}(\phi) \subset \overline{\mathcal{J}}_{g,n}(\phi)$ is an effective divisor supported on the locus of sheaves that admit a nonzero global section, but it is not uniquely determined by its support because $\overline{\Theta}(\phi)$ can be nonreduced (see Corollary 19). We define $\overline{\Theta}(\phi)$ using the formalism of the determinant of cohomology, a formalism we use in Section 4 to compute intersection numbers. More precisely, the theta divisor is defined in terms of the cohomology of the following sheaf:

Definition 13. The **universal family of sheaves** F_{uni} on $\overline{\mathcal{J}}_{g,n}^{\mathrm{pre}}(\phi) \times_{\mathcal{M}_{g,n}^{(0)}} \mathcal{C}_{g,n}$ is defined to be the family of ϕ-stable sheaves that corresponds to the identity under the 2-Yoneda Lemma. A sheaf F_{tau} on $\overline{\mathcal{J}}_{g,n}(\phi) \times_{\mathcal{M}_{g,n}^{(0)}} \mathcal{C}_{g,n}$ is defined to be a **tautological family of sheaves** if F_{tau} is the pullback $(\sigma \times 1)^* F_{\mathrm{uni}}$ of the universal sheaf for some section σ of the natural morphism $\overline{\mathcal{J}}_{g,n}^{\mathrm{pre}}(\phi) \to \overline{\mathcal{J}}_{g,n}(\phi)$.

Concretely F_{tau} is a $\overline{\mathcal{J}}_{g,n}(\phi)$-flat family of rank 1, torsion-free sheaves on $\overline{\mathcal{J}}_{g,n}(\phi) \times_{\mathcal{M}_{g,n}^{(0)}} \mathcal{C}_{g,n}$ such that the restriction to the fiber of $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ over a point $(C, p_1, \ldots, p_n; F) \in \mathcal{M}_{g,n}^{(0)}$ is isomorphic to F.

Lemma 13. *The rigidification morphism $\overline{\mathcal{J}}^{pre}_{g,n}(\phi) \to \overline{\mathcal{J}}_{g,n}(\phi)$ admits a section. In particular, $\overline{\mathcal{J}}_{g,n}(\phi)$ admits a tautological family F_{tau}.*

Proof. A section is defined by rigidifying sheaves along the marking p_1. More formally, consider the morphism $\overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi) \to \overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi)$ that sends a tuple $(C/T, p_1, \dots, p_n; F)$ to $(C/T, p_1, \dots, p_n; F \otimes (\pi_T)^*(p_1^*(F)^{-1}))$. Here $\pi_T: C \to T$ is the structure morphism. To see this morphism is well-defined, observe $F \otimes \pi_T^*(p_1^*(F)^{-1})$ is a flat family ϕ-stable sheaves because this sheaf is Zariski locally isomorphic to F over T (as $p_1^*(F)$ is a line bundle). Furthermore, $\overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi) \to \overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi)$ has the property that the image of $\mathbb{G}_m(T) \subset \text{Aut}(C, p_1, \dots, p_n; F)$ is mapped to the identity in $\text{Aut}(C, p_1, \dots, p_n; F \otimes \pi_T^* p_1^*(F)^{-1})$ (a scalar $g \in \mathbb{G}_m(T)$ acts by g on F, by g^{-1} on $\pi_T^* p_1^*(F)^{-1}$, so by $gg^{-1} = 1$ on the tensor product). By the universal property of rigidification the morphism $\overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi) \to \overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi)$ factors as $\overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi) \to \overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi)$, and $\overline{\mathcal{J}}_{g,n}(\phi) \to \overline{\mathcal{J}}^{\text{pre}}_{g,n}(\phi)$ defines the desired section. ☐

Remark 10. The tautological family F_{tau} is not uniquely determined. Given a tautological family F_{tau} and a line bundle L on $\overline{\mathcal{J}}_{g,n}(\phi)$, $F_{\text{tau}} \otimes \pi^*(L)$ is also a tautological family. However, every tautological family is of the form $F_{\text{tau}} \otimes \pi^* L$ for some line bundle L on $\overline{\mathcal{J}}_{g,n}(\phi)$ by the Seesaw theorem.

We now construct the theta divisor as the determinant of the cohomology of F_{tau}. Recall the more general construction of the determinant of an element of the derived category. Generalizing earlier work with Mumford, Knudsen proved that the rule that assigns to a bounded complex $\mathcal{E}$ of vector bundles on $\overline{\mathcal{J}}_{g,n}(\phi)$ the line bundle $\det(\mathcal{E}) := \otimes(\wedge^{\max} \mathcal{E}^i)^{(-1)^i}$ extends to a rule that assigns an isomorphism of line bundles to a quasi-isomorphism of perfect complexes [Knu02, Theorem 2.3], so the determinant of an object in the bounded derived category is defined. (See also [Est01, Section 6.1] for a more explicit approach in the special case of a family of curves, the case of current interest.) The derived pushforward $\mathbb{R}\pi_* F_{\text{tau}}$ of a tautological family is an element of the bounded derived category by the finiteness theorem [Ill05, Theorem 8.3.8], so in particular, its determinant $\det(\mathbb{R}\pi_* F_{\text{tau}})$ is defined.

The inverse line bundle $\det(\mathbb{R}\pi_* F_{\text{tau}})^{-1}$ admits a distinguished nonzero global section that is constructed as follows. The morphism $\pi: \mathcal{C}_{g,n} \to \mathcal{M}^{(0)}_{g,n}$ has relative cohomological dimension 1, so $\mathbb{R}\pi_* F_{\text{tau}}$ can be represented by a 2-term complex of vector bundles $\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1$. The generic fiber of this complex computes the cohomology of a degree $g-1$ sheaf, so it has Euler characteristic zero (by the Riemann–Roch formula). We deduce that $\text{rank}\, \mathcal{E}^0 = \text{rank}\, \mathcal{E}^1$, and so the top exterior power $\det(d) := \wedge^{\max}(d)$ is a global section of

$$\begin{aligned} \mathscr{H}om\left(\det \mathcal{E}^0, \det \mathcal{E}^1\right) &= \left(\det \mathcal{E}^0\right)^{-1} \otimes \det \mathcal{E}^1 \\ &= \det\left(\mathbb{R}\pi_* F_{\text{tau}}\right)^{-1}. \end{aligned}$$

A direct computation shows that $\det(d) \in H^0(\mathcal{M}^{(0)}_{g,n}, \det(\mathbb{R}\pi_* F_{\text{tau}})^{-1})$ is independent of the choice of complex $\mathcal{E}$ (i.e. that $\det(d)$ is preserved by isomorphisms induced by quasi-isomorphisms; see [Est01, Observation 43]).

The line bundle $\det(\mathbb{R}\pi_* F_{\text{tau}})$ is uniquely determined even though F_{tau} is not:

Lemma 14. *If F_{tau} and G_{tau} are two tautological families on $\overline{\mathcal{J}}_{g,n}(\phi)$, then*

$$\det(\mathbb{R}\pi_* F_{tau}) = \det(\mathbb{R}\pi_* G_{tau}),$$

and this identification identifies

$$\det(d) \in H^0(\mathcal{M}_{g,n}^{(0)}, \det(\mathbb{R}\pi_* F_{tau})^{-1})$$

with

$$\det(e) \in H^0(\mathcal{M}_{g,n}^{(0)}, \det(\mathbb{R}\pi_* G_{tau})^{-1}).$$

Proof. By Remark 10, $G_{\text{tau}} = F_{\text{tau}} \otimes \pi^*(M)$ for some line bundle M on $\overline{\mathcal{J}}_{g,n}(\phi)$, so the result follows from the projection property of the determinant [Est01, Proposition 44(3)]. □

Definition 14. The **theta divisor** $\overline{\Theta}(\phi) \subset \overline{\mathcal{J}}_{g,n}(\phi)$ is the effective Cartier divisor defined by $(\det(\mathbb{R}\pi_* F_{\text{tau}})^{-1}, \det(d))$ for $\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1$ a 2-term complex of vector bundles that represents $\mathbb{R}\pi_* F_{\text{tau}}$. The **theta divisor Chow class** $\overline{\theta}(\phi) \in \mathrm{A}^1(\overline{\mathcal{J}}_{g,n}(\phi))$ is the fundamental class of $\overline{\Theta}(\phi)$.

We conclude this section by describing some of the properties of $\overline{\Theta}(\phi)$.

Lemma 15. *The theta divisor $\overline{\Theta}(\phi)$ is supported on the locus of points $(C, p_1, \ldots, p_n; F) \in \overline{\mathcal{J}}_{g,n}(\phi)$ with $H^0(C, F) \neq 0$.*

Proof. Fix a 2-term complex of vector bundles $\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1$ that represents $\mathbb{R}\pi_* F_{\text{tau}}$, so that $\overline{\Theta}(\phi) = \{\det(d) = 0\}$. Given a point $(C, p_1, \ldots, p_n; F)$, write

$$\mathcal{E} \otimes k(\text{point}) := \mathcal{E}^0 \otimes k(\text{point}) \xrightarrow{d \otimes 1} \mathcal{E}^1 \otimes k(\text{point})$$

for the fiber of $\mathcal{E}^0 \to \mathcal{E}^1$ at $(C, p_1, \ldots, p_n; F)$. The point $(C, p_1, \ldots, p_n; F)$ lies in $\overline{\Theta}(\phi)$ if and only if the complex $\mathcal{E} \otimes k(\text{point})$ has nonzero cohomology, and because the formation of $\mathbb{R}\pi_* F_{\text{tau}}$ commutes with base change [Ill05, Theorem 8.3.2], the cohomology groups of $\mathcal{E} \otimes k(\text{point})$ are $H^0(C, F)$ and $H^1(C, F)$. □

Next we characterize when $\overline{\Theta}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat.

Lemma 16. *If $\mathcal{P} \subset V_{g,n}^{(0)}$ is a stability polytope, then for $\phi \in \mathcal{P}$, the natural projection $\overline{\Theta}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat if and only if $\phi_{can} \in \overline{\mathcal{P}}$.*

Proof. Since $\overline{\Theta}(\phi)$ is an effective divisor on $\overline{\mathcal{J}}_{g,n}(\phi)$ and $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat, $\overline{\Theta}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ is flat if and only if $\overline{\Theta}(\phi)$ does not contain a fiber of $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$. The lemma thus follows from Lemma 15 and [Bea77, Lemma (2.1)]. □

3.4. **Stability Conditions: Concluding Remarks.** We conclude with some remarks, beginning with remarks about the families $\overline{\mathcal{J}}_{g,n}(\phi)$ and their relation to families already existing in the literature. By definition $\overline{\mathcal{J}}_{g,n}(\phi)$ is the moduli space of ϕ-semistable rank 1, torsion-free sheaves. Our definition of ϕ-semistability, Definition 5, is an extension of the definition given by Oda–Seshadri [OS79] (for degree 0 rank 1, torsion-free sheaves on a single nodal curve), and our proof of Proposition 11 shows that ϕ-semistability can be (non-canonically) identified with slope semistablity in the sense of [Sim94]. The authors expect that $\overline{\mathcal{J}}_{g,n}(\phi)$ can alternatively be constructed as a family of quasi-stable compactified Jacobians in the sense of Esteves [Est01].

Earlier Melo constructed a compactified universal Jacobian over $\overline{\mathcal{M}}_{g,n}$ in [Mel11]. Her compactification is different from the ones studied in this paper as e.g. it is not always a Deligne–Mumford stack (as the hypothesis to [Mel11, Proposition 8.3] fails; her compactification is also not a moduli stack of torsion-free sheaves on stable curves, but the authors expect one can identify it with such a stack by an argument similar to [Pan96, Theorem 10.3.1]). Melo's paper builds upon a large body of work on constructing compactifications over $\overline{\mathcal{M}}_{g,0}$ [Cap94, Pan96, Jar00, Cap08b, Mel09].

The difference between the different compactifications over $\overline{\mathcal{M}}_{g,n}$ is somewhat subtle. To describe the difference, fix a stable marked curve $(C, p_1, \dots, p_n) \in \Delta_{i,S}$ that has two k-smooth irreducible components and assume $i, g-i > 0$ (so C does not have a rational tail) and then examine the corresponding fiber $\overline{J}_C$ of $\overline{\mathcal{J}} \to \mathcal{M}_{g,n}^{(0)}$ for $\overline{\mathcal{J}} \to \mathcal{M}_{g,n}^{(0)}$ a family extending the universal Jacobian. For most extensions, $\overline{J}_C$ is isomorphic to the product $J^+ \times J^-$ of the Jacobians of the irreducible components C^+, C^- of C. There are, however, many ways of interpreting this scheme as a moduli space of sheaves: For any pair of integers (d_+, d_-) with $d_+ + d_- = g-1$, we can extend $\mathcal{J}_{g,n}$ by taking $\overline{J}_C$ to be the moduli space of bidegree (d_+, d_-) line bundles on C and then restriction defines an isomorphism $\overline{J}_C \cong J^{d_+} \times J^{d_-}$ with the product of the moduli space of degree d_+ line bundles on C^+ and the moduli space of degree d_- line bundles on C^-. (This moduli space is, for example, the fiber of $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$ for a suitably chosen ϕ.)

We do not study the moduli space $\overline{J}_C(\phi_{\text{can}})$ associated to the canonical parameter in this paper, but this moduli space has frequently been studied in the literature; by [Pan96, Theorem 10.3.1] this is the moduli space constructed in [Cap94, Cap08b], and it plays a distinguished role in the study of degenerations of abelian varieties (see Section 5.2). The moduli space $\overline{J}_C(\phi_{\text{can}})$ can be described as follows. There are three types of ϕ_{can}-semistable sheaves, all of which are strictly semistable: line bundles of bidegree $(i-1, g-i)$, line bundles of bidegree $(i, g-i-1)$, and sheaves that are the direct image of a line bundle of bidegree $(i-1, g-i-1)$ on the normalization. Restriction again defines an isomorphism between the coarse moduli space $\overline{J}_C(\phi_{\text{can}})$ (in the sense of [Sim94]) and $J^{i-1} \times J^{g-i-1}$, so $\overline{J}_C(\phi_{\text{can}})$ is (non-canonically) isomorphic to $\overline{J}_C(\phi)$ for any general ϕ, but this isomorphism cannot be chosen in a way that identifies moduli functors. Furthermore, while the scheme $\overline{J}_C(\phi)$ is a fine moduli space, $\overline{J}_C(\phi_{\text{can}})$ is naturally the coarse space of an algebraic stack with $\mathbb{G}_m$ stabilizers.

There is one important family $\overline{\mathcal{J}}$ with the property that $\overline{J}_C$ is not isomorphic to $J^+ \times J^-$: the family constructed by Caporaso in [Cap94] (over the moduli space of unmarked curves $\overline{\mathcal{M}}_{g,0}$, a space we do not consider here). That family has the property that $\overline{J}_C$ is not $J^+ \times J^-$ but rather the quotient $J^+ \times J^- / \operatorname{Aut}(C)$. The appearance of this quotient is related to stack-theoretic issues; in loc. cit. $\overline{\mathcal{J}}$ is a family over the coarse moduli scheme of the moduli stack $\overline{\mathcal{M}}_{g,0}$ rather than over the stack itself.

For the families we construct in this paper, the fiber $\overline{J}_C$ is isomorphic to a product of Jacobians, and different families only differ on the level of moduli functors. The authors expect this is an artifact of the fact that we study extensions of $\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}$ to a family over $\mathcal{M}_{g,n}^{(0)}$ rather than all of $\overline{\mathcal{M}}_{g,n}$, for examples in [MRV14] suggest that there are many different schemes that extend the universal Jacobian to a family of moduli spaces over all of $\overline{\mathcal{M}}_{g,n}$.

This brings us to the second topic of discussion: the stability space $V_{g,n}^{(0)}$. We have defined $V_{g,n}^{(0)}$ so that it controls families over $\mathcal{M}_{g,n}^{(0)}$. A consequence of Corollary 19 in Section 4 is that the decomposition of $V_{g,n}^{(0)}$ defined by the variation of the theta divisor *essentially* coincides with the stability polytope decomposition, the only difference being that the theta divisor is constant on all the (finitely many) polytopes that contain ϕ_{can} in their closures (a consequence of Lemma 16).

The authors believe that $\mathcal{M}_{g,n}^{(0)}$ is the largest open substack $\mathcal{W} \subset \overline{\mathcal{M}}_{g,n}$ that is a union of topological strata, with the property that the different theta divisors are essentially in bijection with the different extensions of $\mathcal{J}_{g,n}$ to a family over $\mathcal{W}$.

4. Wall-Crossing formula for the Theta divisor

In this section we restrict our attention to nondegenerate stability parameters $\phi \in V_{g,n}^{(0)}$, and study how the theta divisor class $\overline{\theta}(\phi)$ varies with ϕ by proving a wall-crossing formula. The main result of this section is Equation (22), which we prove by applying the Grothendieck-Riemann-Roch theorem to a test curve.

In the previous section we defined a stability space $V_{g,n}^{(0)}$ (Definition 5); for any $\phi \in V_{g,n}^{(0)}$ we then defined a universal ϕ-compactified Jacobian $\overline{\mathcal{J}}_{g,n}(\phi)$ over $\mathcal{M}_{g,n}^{(0)}$ (Definition 12) and a theta divisor $\overline{\Theta}(\phi) \subset \overline{\mathcal{J}}_{g,n}(\phi)$ (Definition 14). In Definition 8 we endowed the stability space $V_{g,n}^{(0)}$ with a stability polytope decomposition. In the paragraph below we prove that the Picard groups of $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and of $\overline{\mathcal{J}}_{g,n}(\phi_2)$ are isomorphic for nondegenerate $\phi_1, \phi_2 \in V_{g,n}^{(0)}$. In fact, the isomorphism is induced by an isomorphism between the moduli stacks $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$, and Lemma 10 prescribes a distinguished choice of such an isomorphism.

Indeed, for every pair of stability polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset V_{g,n}^{(0)}$, by Lemma 10 there exists a unique element $L(\mathcal{P}_1, \mathcal{P}_2)$ in the subgroup of $\operatorname{Pic}^0(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}^{(0)})$ generated by the components $\mathcal{O}(\mathcal{C}_{i,S}^{\pm})$ such that the rule $F \mapsto F \otimes L(\mathcal{P}_1, \mathcal{P}_2)$ defines an isomorphism $\overline{\mathcal{J}}_{g,n}(\phi_1) \to \overline{\mathcal{J}}_{g,n}(\phi_2)$ for all $\phi_1 \in \mathcal{P}_1$ and $\phi_2 \in \mathcal{P}_2$. In this section we will always identify $\overline{\mathcal{J}}_{g,n}(\phi_1)$ and $\overline{\mathcal{J}}_{g,n}(\phi_2)$ using the isomorphism $L(\mathcal{P}_1, \mathcal{P}_2)$. Our main result is a formula for the difference $\overline{\theta}(\phi_2) - \overline{\theta}(\phi_1)$.

We describe this difference between theta divisors by fixing a stability wall (or facet) H, and then describing the difference between the theta classes associated to two stability polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ that have H as a common facet. By Lemma 6 a wall $H = H(i, S, d)$ in the stability space $V_{g,n}^{(0)}$ is determined by a stable graph with two vertices and one edge $\Gamma(i, S)$, and by an integer d. The polytope $\mathcal{P}_2$ is a translate of $\mathcal{P}_1$, and we fix the convention that

$$\mathcal{P}_2(i, S) = \mathcal{P}_1(i, S) + (1, -1).$$

In more concrete terms, for a general fiber over $\Delta_{i,S}$, the ϕ-stable sheaves are the line bundles of bidegree

$$(d-1, g-d) \text{ when } \phi \in \mathcal{P}_1, \text{ and } (d, g-1-d) \text{ when } \phi \in \mathcal{P}_2.$$

We can now formulate the wall-crossing formula for the theta divisor class.

Theorem 17. *Let ϕ_1, ϕ_2 be nondegenerate stability parameters that belong to stability polytopes $\mathcal{P}_1, \mathcal{P}_2$ of $V_{g,n}^{(0)}$ whose common facet is the wall $H(\Gamma(i,S),d) =: H(i,S,d)$. Then*

$$\begin{aligned} \overline{\theta}(\phi_2) - \overline{\theta}(\phi_1) &= \left(\left\lfloor \phi_2^+(i,S) + \frac{1}{2} \right\rfloor - i \right) \cdot \delta_{i,S} \\ &= (d-i) \cdot \delta_{i,S}. \end{aligned} \tag{22}$$

(As is customary, we have written $\delta_{i,S}$ for the pullback of the boundary divisor class along $\overline{\mathcal{J}}_{g,n} \to \mathcal{M}_{g,n}^{(0)}$).

Proof. Choosing tautological bundles $F_{\text{tau}}(\phi_1)$ and $F_{\text{tau}}(\phi_2)$ as in Lemma 13, we have

$$F_{\text{tau}}(\phi_2) \cong F_{\text{tau}}(\phi_1) \otimes \mathcal{O}(\mathcal{C}_{i,S}^-).$$

By Definition 14, the left-hand side of (22) is the first Chern class of the line bundle

$$L := \left(\det(\mathbb{R}\pi_* F_{\text{tau}}(\phi_1) \otimes \mathcal{O}(\mathcal{C}_{i,S}^-))\right)^{-1} \otimes \det(\mathbb{R}\pi_* F_{\text{tau}}(\phi_1)). \tag{23}$$

We claim that the line bundle L is the pullback of $\mathcal{O}(\Delta_{i,S})^{\otimes c}$ for some $c \in \mathbb{Z}$. Indeed, over the complement of $\Delta_{i,S}$ the restriction of $\mathcal{O}(\mathcal{C}_{i,S}^-)$ is trivial. Since the formation of the determinant of cohomology commutes with base change, the restriction of L to $\overline{\mathcal{J}}_{g,n} - \Delta_{i,S}$ is also trivial, which implies our claim.

The integer c is determined by computing the other two integers in the equality

$$c \cdot \deg\left(\mathcal{O}(\Delta_{i,S})_{|T}\right) = \deg L_{|T}. \tag{24}$$

Let $(\pi_T: C \to T, p_1, \dots, p_n)$ be a **test curve** for $\mathcal{M}_{g,n}^{(0)}$: the pullback to a k-smooth curve $T \to \mathcal{M}_{g,n}^{(0)}$ of the universal curve $\pi: \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}^{(0)}$ and of the universal sections. Every such $T \to \mathcal{M}_{g,n}^{(0)}$ lifts to a morphism $T \to \overline{\mathcal{J}}_{g,n}(\phi_1)$; equivalently there exists a family of π_T-fiberwise ϕ_1-stable sheaf F on C. This is a consequence of the existence of a section of the forgetful morphism $\overline{\mathcal{J}}_{g,n}(\phi_1) \to \mathcal{M}_{g,n}^{(0)}$, and we will construct several such sections in the beginning of Section 5, see (39) and (40).

From Proposition 21 below (a Grothendieck-Riemann-Roch calculation) we deduce that the right-hand side of (24) equals

$$\deg L_{|T} = -\deg\left(\pi_{T*}\left(\left(\operatorname{ch} F(\mathcal{C}_{i,S}^-) - \operatorname{ch}(F)\right) \cap \operatorname{td} C\right)\right). \tag{25}$$

In Construction 1 we produce an explicit test curve for $\mathcal{M}_{g,n}^{(0)}$ whose intersection with the divisor $\Delta_{i,S}$ is the class of one point:

$$\deg\left(\mathcal{O}(\Delta_{i,S})_{|T}\right) = 1. \tag{26}$$

We then prove in Lemma 22 that this test curve, the right-hand side of (25) equals

$$\deg L_{|T} = -\deg\left(F_{|\mathcal{C}_{i,S}^-}\right) + (g-i) = -(g-1-d+1) + (g-i) = (d-i). \tag{27}$$

Combining Equation (24) with (26) and (27) gives $c = d - i$, which concludes the proof of Theorem 17. □

Remark 11. Using the classical Riemann-Roch formula, we can express the coefficient of $\delta_{i,S}$ in Formula (22) as the Euler characteristic of a line bundle.

Writing $C_{i,S} = C^+ \cup C^-$ for a general fiber of $\pi^{-1}(\Delta_{i,S}) \to \Delta_{i,S}$, we have the equalities

$$d - i = \deg(F_{\mathrm{tau}}(\phi_1)_{|C^+}) + 1 - i = \chi(C^+, F_{\mathrm{tau}}(\phi_1)).$$

Formula (22) can therefore be written in the form

$$\overline{\theta}(\phi_2) - \overline{\theta}(\phi_1) = \chi(C^+, F_{\mathrm{tau}}(\phi_1)) \cdot \delta_{i,S} = -\chi(C^-, F_{\mathrm{tau}}(\phi_2)) \cdot \delta_{i,S}. \tag{28}$$

Let us now present some easy corollaries of formula (22). As a consequence of Lemma 16, $\overline{\Theta}(\phi_2)$ equals $\overline{\Theta}(\phi_1)$ when the canonical parameter ϕ_{can} belongs to the closures of the stability polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$. Theorem 17 provides the converse implication.

Corollary 18. *If $\overline{\Theta}(\phi_1)$ equals $\overline{\Theta}(\phi_2)$, then ϕ_{can} belongs to the closures of both $\mathcal{P}_1$ and $\mathcal{P}_2$.*

When ϕ satisfies the conditions of Corollary 18, $\overline{\Theta}(\phi)$ is reduced. In the following corollary we determine all the stability parameters whose associated theta divisor is reduced.

Corollary 19. *If $\phi \in V_{g,n}^{(0)}$ belongs to the stability polytope $\mathcal{P}$, then $\overline{\Theta}(\phi)$ is reduced if and only if there exists a stability polytope $\mathcal{Q}$ such that $\overline{\mathcal{P}} \cap \overline{\mathcal{Q}} \neq \varnothing$ and $\phi_{can} \in \overline{\mathcal{Q}}$.*

Proof. If we let $\Theta(\phi)_{\mathrm{red}}$ be the reduced structure on the theta divisor, we have the inclusion of subschemes $\Theta(\phi)_{\mathrm{red}} \subseteq \Theta(\phi)$. Arguing as in the first paragraph of the proof of Theorem 17, we see that $\Theta(\phi)$ consists of an irreducible component dominant over $\mathcal{M}_{g,n}^{(0)}$, and of other components supported over the inverse image of the boundary divisors $\Delta_{i,S}$. The first component is isomorphic to $\Theta(\phi_0)$ for $\phi_0 \in \mathcal{Q}$, which is reduced by Lemma 16. We can then write an equality of divisor classes

$$\theta(\phi) = \theta(\phi)_{\mathrm{red}} + \sum_{(i,S)} a_{i,S} \cdot \Delta_{i,S}$$

(where once again we have written $\Delta_{i,S}$ for the pullback of the boundary divisor class along $\overline{\mathcal{J}}_{g,n} \to \mathcal{M}_{g,n}^{(0)}$). If $\Theta(\phi)$ is reduced, then so are all its irreducible components, which implies that all coefficients $a(i,S)$ are either 0 or 1. The converse implication is provided in Lemma 20. It follows from our main Theorem 17 that the coefficients $a_{i,S}$ are either 0 or 1 precisely when $\overline{\mathcal{P}}$ and $\overline{\mathcal{Q}}$ have a common facet, where $\overline{\mathcal{Q}}$ is any of the polytopes containing ϕ_{can} in its closure. □

In the proof of Corollary 19 we use following (probably well-known) lemma, for which we provide a proof as we could not find a suitable reference.

Lemma 20. *Let $\overline{\mathcal{X}}$ be a smooth proper Deligne-Mumford stack, and D_1 and D_2 be two effective divisors with D_1 a closed substack of D_2. Then the inclusion induces an isomorphism of D_1 and D_2 if and only if the divisor classes $[D_1]$ and $[D_2]$ coincide.*

Proof. First reduce to the setting of divisors on a projective scheme by picking an étale cover $\overline{\mathcal{M}} \to \overline{\mathcal{X}}$ with $\overline{\mathcal{M}}$ smooth and projective. We have an exact sequence of coherent sheaves

$$0 \to \mathcal{O}_{D_2}(-D_1) \to \mathcal{O}_{D_2} \to \mathcal{O}_{D_1} \to 0.$$

Now pick an ample line bundle A on $\overline{\mathcal{M}}$ and consider the Hilbert polynomials $p_1(t)$ and $p_2(t)$ respectively associated to D_1 and D_2. The polynomial p_i has degree $\dim(\overline{\mathcal{M}})-1$ and leading term the degree of D_i (computed with respect to A). Since D_1 is linearly equivalent to D_2 by hypothesis, $\mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}$ have the same degree, and so p_2-p_1 has degree strictly less than $\dim(\overline{\mathcal{M}})-1$. By additivity p_2-p_1 equals the Hilbert polynomial of $\mathcal{O}_{D_2}(-D_1)$, and we conclude that this last Hilbert polynomial has degree strictly less than $\dim(\overline{\mathcal{M}})-1$.

This is only possible if $\mathcal{O}_{D_2}(-D_1)$ equals zero. Indeed, $\mathcal{O}_{D_2}(-D_1)$ is locally principal, so if $\mathcal{O}_{D_2}(-D_1)$ was nonzero, then its support would have dimension $\dim(\overline{\mathcal{M}})-1$. Since $\mathcal{O}_{D_2}(-D_1)=0$, the inclusion of D_1 in D_2 is an isomorphism. □

We conclude this section by providing the proof of the auxiliary results that we used to prove Theorem 17.

Proposition 21. *Let $(\pi_T\colon C\to T, p_1,\dots,p_n,F)$ be a test curve for $\overline{\mathcal{J}}_{g,n}(\phi)$. Then the following equality*

$$(c_1\left(\mathbb{R}\pi_* F(\mathcal{C}^-_{i,S})\right)-c_1(\mathbb{R}\pi_*F))\cap[T]=\pi_{T*}\left(\left(\operatorname{ch}F(\mathcal{C}^-_{i,S})-\operatorname{ch}(F)\right)\cap\ \operatorname{td}C\right)$$

holds in the Chow group of 0*-cycles on T.*

Proof. The 0-th and 1-st higher pushforwards of F and $F(\mathcal{C}^-_{i,S})$ under π_T are sheaves of the same rank. Indeed, taking higher pushforwards commutes with base change, and the 0-th and 1-st cohomology of F and $F(\mathcal{C}^-_{i,S})$ on the fiber of a geometric point in T have the same dimension by the Riemann-Roch formula for curves, since the fiberwise degree is $g-1$. Therefore we have that both the degree-0 Chern characters

$$\operatorname{ch}_0(\mathbb{R}\pi_*F),\quad \operatorname{ch}_0(\mathbb{R}\pi_*F(\mathcal{C}^-_{i,S}))$$

vanish, so we deduce the following equality in the Chow group of 0-cycles on T:

$$c_1\left(\mathbb{R}\pi_*F(\mathcal{C}^-_{i,S})\right)-c_1(\mathbb{R}\pi_*F))\cap[T]=\left(\operatorname{ch}\mathbb{R}\pi_*F(\mathcal{C}^-_{i,S})-\operatorname{ch}\mathbb{R}\pi_*F\right)\cap\ \operatorname{td}T. \tag{29}$$

Applying the Grothendieck-Riemann-Roch formula to π_T, we have

$$\left(\operatorname{ch}\mathbb{R}\pi_*F(\mathcal{C}^-_{i,S})-\operatorname{ch}\mathbb{R}\pi_*F\right)\cap\ \operatorname{td}T=\pi_*\left(\left(\operatorname{ch}F(\mathcal{C}^-_{i,S})-\operatorname{ch}(F)\right)\cap\ \operatorname{td}C\right), \tag{30}$$

and the statement follows by combining Equations (29) and (30). □

Construction 1. For each pair (i,S) as in Definition 2, we construct a test curve $(\pi_T\colon C\to T,p_1,\dots,p_n)$ whose intersection with $\Delta_{i,S}$ is the class of one point, and whose general fiber is in $\Delta_{i,S\smallsetminus\{1\}}$, as shown in Figure 5. (The special cases $i=0$ and $|S|=2$ are left to the scrupulous reader).

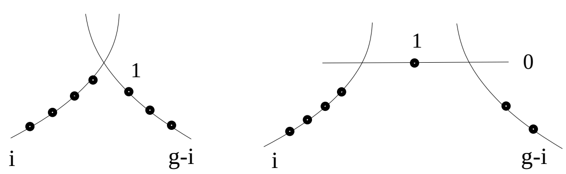

FIGURE 5. The general fiber and the special fiber of the test curve

Fix a general genus $g-i$ marked curve $(T,(S^c\cup\{1\})$ and a general genus i marked curve $(T',S\cup\{\bullet\}\smallsetminus\{1\})$. In $T\times T$ the diagonal intersects the locus

$$T\times\{p_k: k\in S^c\cup\{1\}\}$$

at the points $\{(p_k,p_k): k\in S^c\cup\{1\}\}$, and we define the blow-up of these points to be $\widetilde{C}_1\to T\times T$. We then define $\widetilde{C}_2$ to be $T\times T'$.

The diagonal map $\Delta: T\to T\times T$ induces a morphism $s_1: T\to\widetilde{C}_1$, and we define the morphism $s_2: T\to\widetilde{C}_2$ as the constant $\bullet$ section of the first projection map. We then define C to be the following pushout (which exists by [Fer03, Theorem 5.4])

$$\begin{array}{ccc} T\cup T & \xrightarrow{s_1\cup s_2} & \widetilde{C}_1\cup\widetilde{C}_2 \\ \big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ T & \xrightarrow{\;\;j\;\;} & C. \end{array}$$

Projection onto the first component defines a morphism $\widetilde{C}_1\cup\widetilde{C}_2\to T$ that induces a morphism $\pi_T: C\to T$ by the universal property of pushouts. The morphism π_T inherits pairwise disjoint sections $p_1,\dots,p_n$, whose images lies in the fiberwise k-smooth locus.

The family π_T then defines a morphism $T\to\mathcal{M}^{(0)}_{g,n}$ whose intersection with $\Delta_{i,S}$ is the class of one point (by construction the total space C of the family is k-smooth at the unique node of type $\Delta_{i,S}$).

Lemma 22. *On the test curve $(\pi_T: C\to T,p_1,\dots,p_n)$ defined in Construction 1, we have*

$$\deg\left(\pi_{T*}\left(\left(\operatorname{ch}F(\mathcal{C}^-_{i,S})-\operatorname{ch}(F)\right)\cap\ \operatorname{td}C\right)\right)=\deg\left(L_{|\mathcal{C}^-_{i,S}}\right)-(g-i) \tag{31}$$

Proof. Since we are after the calculation of the degree of a 0-cycle, from now on it will be enough to compute all classes *modulo numerical equivalence.* In the calculations we will follow standard notation: we omit writing fundamental classes, we write (a,b) to denote the ruling on a product, and we write [pt] for the class of a point.

We claim that the Todd class of C equals

$$\operatorname{td}(C)=\nu_*(\operatorname{td}(\tilde{C}_1\times\tilde{C}_2))-j_*(\operatorname{td}(T)). \tag{32}$$

Indeed, the Todd class of the singular variety C is defined in [Ful98, Chapter 18] as $\tau_C(\mathcal{O}_C)$, for τ_C a group homomorphism from the K-theory of coherent sheaves on C to the rational Chow group of C. Formula (32) follows by applying τ_C to the short sequence of sheaves

$$0\to\mathcal{O}_C\to\nu_*(\mathcal{O}_{\tilde{C}_1\times\tilde{C}_2})\to j_*\mathcal{O}_T\to 0,$$

which is exact by the pushout construction.

We denote by ν_1 and ν_2 the restrictions of ν to the two components of the normalization $\tilde{C}=\tilde{C}_1\sqcup\tilde{C}_2$. Applying the formulas for the Todd class of a product and of a blow-up of a k-smooth surface at a point, we compute

$$\begin{cases}\operatorname{td}(T)= & 1+(1-g+i)[\mathrm{pt}],\\ \operatorname{td}(\tilde{C}_1)= & 1+(1-g+i,1-g+i)-\frac{1}{2}\sum_{k\in S^c\cup\{1\}}E_k+(1-g+i)^2[\mathrm{pt}],\\ \operatorname{td}(\tilde{C}_2)= & 1+(1-g+i,1-i)+(1-g+i)(1-i)[\mathrm{pt}].\end{cases} \tag{33}$$

(Here E_k denotes the exceptional fibers of the surface $\tilde{C}_1$.)

The relevant Chern characters on $\tilde{C}_1$ are

$$\begin{cases} \operatorname{ch} \nu_1^* \mathcal{O}(\mathcal{C}_{i,S}^+) = & 1 + E_1 - \frac{1}{2}[\mathrm{pt}], \\ \operatorname{ch} \nu_1^* \mathcal{O}(\mathcal{C}_{i,S}^-) = & 1 + (1,0) - E_1 - \frac{1}{2}[\mathrm{pt}]; \end{cases} \tag{34}$$

and the relevant Chern characters on $\tilde{C}_2$ are

$$\begin{cases} \operatorname{ch} \nu_2^* \mathcal{O}(\mathcal{C}_{i,S}^+) = & 1 + (1,0), \\ \operatorname{ch} \nu_2^* \mathcal{O}(\mathcal{C}_{i,S}^-) = & 1. \end{cases} \tag{35}$$

To calculate the degree on the left-hand side of (31), we compute

$$\begin{aligned} \deg\left(\left(\operatorname{ch} \nu_1^* F(\mathcal{C}_{i,S}^-) - \operatorname{ch} \nu_1^* F\right) \cap \operatorname{td} \tilde{C}_1\right) &= (1 - g + i) - \frac{1}{2} + \deg(F_{|\mathcal{C}_{i,S}^-}) - \frac{1}{2} \\ &= \deg(F_{|\mathcal{C}_{i,S}^-}) - (g - i), \end{aligned} \tag{36}$$

$$\deg\left(\left(\operatorname{ch} \nu_2^* F(\mathcal{C}_{i,S}^-) - \operatorname{ch} \nu_2^* F\right) \cap \operatorname{td} \tilde{C}_2\right) = 0, \tag{37}$$

$$\deg\left(\left(\operatorname{ch} j^* F(\mathcal{C}_{i,S}^-) - \operatorname{ch} j^* F\right) \cap \operatorname{td} T\right) = 0; \tag{38}$$

where the last two expressions vanish because the curve $\mathcal{C}_{i,S}^-$ has empty intersection with $\nu_2(\tilde{C}_2)$ and with $j(T)$.

Altogether, taking (36) + (37) − (38), we find that the degree on the left-hand side of (31) equals $\deg\left(F_{|\mathcal{C}_{i,S}^-}\right) - (g - i)$. This concludes the proof of Lemma 22. $\square$

5. Pullback of the theta divisor to $\overline{\mathcal{M}}_{g,n}$

In this section we study the pullback of the theta divisor to the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ (or $\overline{\mathcal{M}}_{g,n}^{(0)}$), and compare our results with the existing literature. As in Section 4, we will only work with nondegenerate stability parameters $\phi \in V_{g,n}^{(0)}$.

Let $\vec{d} = (d_1, \ldots, d_n)$ be a vector of integers such that $\sum_{j=1}^n d_j = g - 1$. For any such vector the rule

$$(\pi_T : C \to T, p_1, \ldots, p_n) \mapsto (\pi_T : C \to T, p_1, \ldots, p_n;\ \mathcal{O}_C(\mathcal{D})), \tag{39}$$

with $\mathcal{D} := d_1 p_1 + \ldots + d_n p_n$, defines a section of the forgetful map $\mathcal{J}_{g,n} \to \mathcal{M}_{g,n}$.

For $\phi_{\vec{d}}$ the nondegenerate parameter defined by

$$\phi_{\vec{d}}(i,S) = (\phi^+(i,S), \phi^-(i,S)) := \left(\sum_{j \in S} d_j, g - 1 - \sum_{j \in S} d_j\right) =: (d_S, g - 1 - d_S),$$

the family of line bundles $\mathcal{O}_C(\mathcal{D})$ is fiberwise $\phi_{\vec{d}}$-stable and the rule (39) defines a section $s_{\vec{d}}$ of $\overline{\mathcal{J}}_{g,n}(\phi_{\vec{d}}) \to \mathcal{M}_{g,n}^{(0)}$.

More generally, for any nondegenerate stability parameter $\phi \in V_{g,n}^{(0)}$, we define the divisor

$$\mathcal{D}(\phi) := d_1 p_1 + \ldots + d_n p_n + \sum_{i,S} \left(d_S - \left\lfloor \phi^+(i,S) + \frac{1}{2} \right\rfloor\right) \cdot \mathcal{C}_{i,S}^+. \tag{40}$$

The family of line bundles $\mathcal{O}_C(\mathcal{D}(\phi))$ is fiberwise ϕ-stable by construction, and the rule (39) defines a section[1] $s_{\vec{d}}$ of $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}_{g,n}^{(0)}$. This section is the unique section extending (3), the section out of $\mathcal{M}_{g,n}$ discussed in the introduction, because $\overline{\mathcal{J}}_{g,n}(\phi)$ is separated. We then define $D_{\vec{d}}(\phi) \coloneqq s_{\vec{d}}^{-1}(\theta(\phi))$.

In the following we compute the pullback of the theta class via $s_{\vec{d}}$. Observe that the pullback along $s_{\vec{d}}$ induces a well-defined group homomorphism $\mathrm{Pic}(\overline{\mathcal{J}}_{g,n}) \to \mathrm{Pic}(\mathcal{M}_{g,n}^{(0)})$. (And, as we already observed, the latter is isomorphic to $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ because $\overline{\mathcal{M}}_{g,n} \smallsetminus \mathcal{M}_{g,n}^{(0)}$ has codimension 2).

Recall that the integral Picard group of $\overline{\mathcal{M}}_{g,n}$ is generated (freely when $g \geq 3$) by the first Chern class of the Hodge bundle λ, the first Chern classes of the cotangent line bundles to the j-th marking ψ_j, the boundary strata classes $\delta_{i,S}$ and δ_{irr}.

Theorem 23. *The pullback of $\overline{\theta}(\phi_{\vec{d}})$ from $\overline{\mathcal{J}}_{g,n}(\phi_{\vec{d}})$ to $\overline{\mathcal{M}}_{g,n}$ is given by*

$$s_{\vec{d}}^{*}\, \overline{\theta}(\phi_{\vec{d}}) = -\lambda + \sum_{j=1}^{n} \binom{d_j + 1}{2} \cdot \psi_j. \tag{41}$$

More generally, for any nondegenerate $\phi \in V_{g,n}^{(0)}$, we obtain the equality

$$s_{\vec{d}}^{*}\, \overline{\theta}(\phi) = -\lambda + \sum_{j=1}^{n} \binom{d_j + 1}{2} \cdot \psi_j + \sum_{i,S} \left(\binom{\lfloor \phi^{+}(i,S) + \frac{1}{2} \rfloor - i + 1}{2} - \binom{d_S - i + 1}{2} \right) \cdot \delta_{i,S}. \tag{42}$$

Proof. Assuming (41) holds, Formula (42) follows by the wall-crossing Formula (22).

We prove equality (41). We define $\mathcal{D}$ to the effective divisor $\sum_{j=1}^{n} d_j p_j$ in $\mathcal{C}_{g,n}$. As we observed earlier, the line bundle $\mathcal{O}(\mathcal{D})$ is fiberwise $\phi_{\vec{d}}$-stable. We have

$$\begin{aligned} s_{\vec{d}}^{*}\, \overline{\theta}(\phi_{\vec{d}}) &= -s_{\vec{d}}^{*}\, c_1 \mathbb{R}\pi_{*}(F_{\mathrm{tau}}) = -c_1(\mathbb{R}\pi_{*}\mathcal{O}(\mathcal{D})) \\ &= -\left[\mathrm{ch}\, \mathbb{R}\pi_{*}(\mathcal{O}(\mathcal{D})) \cap \mathrm{td}\left(\mathcal{M}_{g,n}^{(0)}\right)\right]_{\mathrm{codim}=1} \\ &= -\pi_{*}\left[\mathrm{ch}\, \mathcal{O}(\mathcal{D}) \cap \mathrm{td}\left(\mathcal{C}_{g,n}\right)\right]_{\mathrm{codim}=2} \\ &= \pi_{*}\left[-\frac{\mathcal{D}^2}{2} + \mathcal{D} \cdot \frac{K_{\mathcal{C}_{g,n}}}{2} - \mathrm{td}_2(\mathcal{C}_{g,n}) \right], \end{aligned} \tag{43}$$

where we applied the definition of theta divisor, the fact that $\mathrm{ch}_0(\mathbb{R}\pi_{*}\mathcal{O}(\mathcal{D}))$ equals zero, and then the Grothendieck-Riemann-Roch formula for stacks (see e.g. [Edi13, Theorem 3.5]).

The first term in (43) equals

$$-\pi_{*}\left(\frac{\mathcal{D}^2}{2}\right) = \frac{1}{2} \sum_{j=1}^{n} d_j^2 \psi_j, \tag{44}$$

because two different sections p_j and p_k are by definition disjoint, and by the very definition of the ψ-classes:

$$\psi_j \coloneqq -\pi_{*}(p_j^2).$$

[1]In fact, the section $s_{\vec{d}} = s_{\vec{d}}(\phi)$ depends on ϕ, but we will omit to write this dependence to keep the notation simple.

To compute the second and third terms in (43), we identify the universal curve $\mathcal{C}_{g,n}$ with $\overline{\mathcal{M}}_{g,n+1}$. The canonical class equals

$$K = K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta, \text{ where } \delta := \delta_{irr} + \sum \delta_{i,S} \text{ and } \psi := \sum_{j=1}^{n} \psi_j.$$

Using the pushforward formulas

$$\begin{aligned}
\pi_*(p_j \cdot \lambda) &= \lambda, \\
\pi_*(p_j \cdot \psi_k) &= \begin{cases} 0 & \text{when } j = k, \\ \psi_k & \text{when } j \neq k, \end{cases} \\
\pi_*(\psi_j \cdot \delta_{irr}) &= \delta_{irr}, \\
\pi_*(p_j \cdot \delta_{i,S}) &= \begin{cases} \delta_{i,S} & \text{when } \{p_j, p_{n+1}\} \subseteq S \text{ or } \{p_j, p_{n+1}\} \subseteq S^c, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

(where $\delta_{0,\{j\}}$ is interpreted as $-\psi_j$ in the last formula), the second term in (43) becomes

$$\pi_*\left([\mathcal{D}] \cdot \frac{K_{\mathcal{C}_{g,n}}}{2}\right) = \frac{13}{2}(g-1) \cdot \lambda + \sum_{j=1}^{n} \frac{g-1+d_j}{2} \cdot \psi_j - (g-1) \cdot \delta. \tag{45}$$

Finally, the third term equals

$$-\pi_*(\mathrm{td}_2(\mathcal{C}_{g,n})) = -\left(\frac{g-1}{2} \cdot (13\lambda + \psi - 2\delta) + \lambda\right). \tag{46}$$

Indeed $\mathrm{td}_2 = \frac{K^2+c_2}{12}$, and we read the formula for c_2 in [Bin05, page 765]. (Note that the formula for the second Chern class appears with an error in the coefficient of κ_2, which should be $-\frac{1}{2}$. This can be quickly checked by applying the Grothendieck-Riemann-Roch formula to the sheaf $\omega_\pi^{\otimes 2}(p_1 + \ldots + p_n)$ along the universal curve $\pi: \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$.) The pushforward (46) can then be computed with the aid of the pushforward formulas

$$\begin{aligned}
\pi_*(K^2) = \pi_*(\pi^* K + \omega_\pi) &= 2 \cdot \pi_*(\omega_\pi) \cdot K + \pi_*(\omega_\pi^2) \\
&= 2 \cdot (2g-2) \cdot (13\lambda + \psi - 2\delta) + 12\lambda - \delta, \\
\pi_*(\kappa_2) &= 12\lambda + \psi - \delta, \\
\pi_*(\xi_{irr*}(\psi + \psi)) &= 2 \cdot \delta_{irr}, \\
\pi_*(\xi_{i,S*}(1 \otimes \psi + \psi \otimes 1)) &= \delta_{i,S}, \\
\pi_*(\xi_{0,\{j,n+1\}*}(1 \otimes \psi + \psi \otimes 1)) &= \psi_j.
\end{aligned}$$

(Following the notation from [Bin05], here ξ_{irr} and $\xi_{i,S}$ are the gluing maps, and κ_2 is the Arbarello-Cornalba kappa class).

Plugging the three terms (44) (45) and (46) in equation (43), we deduce (41). □

We now compare our result with pullbacks of the theta divisor that have recently been studied by different authors.

5.1. **The class of Hain.** Hain studied a problem similar to the problem of computing $s^*_{\vec{d}}(\overline{\theta}(\phi))$. He constructed a theta divisor on the moduli space of multidegree 0 line bundles on compact type curves (which is fundamentally different from the theta divisors studied in this paper). First, his construction is different. Hain's construction involves a choice of theta characteristic, uses the formalism of theta functions, and produces a $\mathbb{Q}$-divisor class [Hai13, Section 11.2, page 561]. Second, the pullback of the resulting divisor class differs from the pullbacks of the $\theta(\phi)$'s constructed in this paper. Indeed, in [Hai13, Theorem 11.7], Hain computed the pullback of θ_α by $s_{\vec{d}}$ as:

$$[\overline{D}_{\vec{d}}(\mathrm{Ha})] = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j - \sum_{i,S} \binom{d_S - i + 1}{2} \cdot \delta_{i,S} + \frac{\delta_{irr}}{8} \tag{47}$$

$$= [\overline{D}_{\vec{d}}(\phi_0)] + \frac{\delta_{irr}}{8}, \tag{48}$$

and being a nonintegral Chow class, this never equals $s^*_{\vec{d}}(\overline{\theta}(\phi))$.

The results of this paper illuminate some of the structure of (47). The term $\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j$ is $[\overline{D}_{\vec{d}}(\phi_{\vec{d}})]$, while the term $\sum \binom{d_S-i+1}{2} \cdot \delta_{i,S}$ is a wall-crossing term, the difference between $[\overline{D}_{\vec{d}}(\phi_{\vec{d}})]$ and $[\overline{D}_{\vec{d}}(\phi_0)]$ described by Theorem 17.

Finally, a caution to the reader. Grushevsky–Zakharov gave an alternative proof of (47) in [GZ14, Theorem 2, Equation (3.4)]), and their definition of the theta divisor in [GZ14] is different from the definition in [Hai13]. Over the locus of compact type curves, the theta divisor is defined on [GZ14, page 4053, second paragraph] to be the image of an Abel map out of a symmetric power. It is significant that this is taken as the definition over the locus of compact type curves and not over all of $\overline{\mathcal{M}}_{g,n}$. While the image of the Abel map is a divisor class defined over all of $\overline{\mathcal{M}}_{g,n}$, it is not a divisor class whose pullback is $[\overline{D}_{\vec{d}}(\mathrm{Ha})]$ because the image of the Abel map, being the image of a rational morphism between Deligne-Mumford stacks that are representable over $\overline{\mathcal{M}}_{g,n}$, is an integral Chow class, and as such, its pullback by $s_{\vec{d}}$ cannot equal a nonintegral class such as (47).

5.2. **The stable pairs class.** In the introduction we introduced the divisor $[\overline{D}_{\vec{d}}(\mathrm{SP})]$ that is the pullback of the theta divisor of the unique family of stable semiabelic (or quasiabelian) pairs extending the principally polarized universal Jacobian. Here we describe this extension in greater detail.

Recall that a stable semiabelic pair is a pair $(\overline{P}, D)$ consisting of a (possibly reducible) seminormal projective variety $\overline{P}$ with a suitable action of a semiabelian variety G together with an ample effective divisor $D \subset \overline{P}$ that does not contain a G-orbit [Ale02, Definition 1.1.9]. Stable semiabelic pairs satisfy a stable reduction theorem [Ale02, Theorem 5.7.1] that implies there is, up to isomorphism of pairs, at most one extension of the family of principally polarized Jacobians $(\mathcal{J}_{g,n}/\mathcal{M}_{g,n}, \Theta)$ to a family of stable semiabelic pairs $(\overline{\mathcal{J}}_{g,n}/\mathcal{M}^{(0)}_{g,n}, \overline{\Theta})$.

For $n = 0$ (a case not studied here), Alexeev has proven that this unique extension exists and is realized by the Caporaso–Pandharipande family, the family of compactified Jacobians associated to the degenerate parameter ϕ_{can} [Ale04, Theorem 5.1, Theorem 5.3, Corollary 5.4]. For $n > 0$, the unique extension $(\overline{\mathcal{J}}_{g,n}/\overline{\mathcal{M}}_{g,n}, \overline{\Theta})$ of $\mathcal{J}_{g,n}$ is the pullback of $(\overline{\mathcal{J}}_{g,0}, \overline{\Theta}_{g,0})$ by the forgetful morphism $\mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,0}$.

An alternative description of this extension is provided by the following lemma:

Lemma 24. *For ϕ satisfying the condition from Lemma 16, the restriction of the pair $(\overline{\mathcal{J}}_{g,n}(\phi)/\mathcal{M}^{(0)}_{g,n}, \overline{\Theta}(\phi))$ to the open substack $\mathcal{U} \subset \mathcal{M}^{(0)}_{g,n}$ of stable curves with at most 1 node is a stable semiabelic pair.*

Proof. The main point to prove is that a fiber of $\overline{\Theta}(\phi)|\mathcal{U} \to \mathcal{U}$ is ample and does not contain an orbit of the action of the multidegree 0 Jacobian, and we prove this by directly computing the theta divisor, which has a particularly simple structure. To begin, observe that both $\overline{\mathcal{J}}_{g,n}(\phi)|\mathcal{U} \to \mathcal{U}$ and $\overline{\Theta}(\phi)|\mathcal{U} \to \mathcal{U}$ are flat by Proposition 11 and Lemma 16, so it is enough to fix a marked curve $(C, p_1, \dots, p_n) \in \mathcal{U}$ and prove that the fiber $\overline{\mathcal{J}}_C$ with the effective divisor $\overline{\Theta}_C$ is a stable semiablic variety.

Alexeev has proved quite generally that the compactified Jacobian of a nodal curve is a stable semiabelic variety [Ale04, Theorem 5.1], so to prove the specific pair $(\overline{\mathcal{J}}_C, \overline{\Theta}_C)$ is a stable pair, we need to prove that $\overline{\Theta}_C$ is ample and does not contain an orbit of the action of the moduli space $\mathcal{J}_C$ of multidegree 0 line bundles. There are two cases to consider: when C is irreducible and when C is reducible.

When C is irreducible, $\overline{\Theta}_C$ is ample by [Sou94, Corollary 14] and does not contain a group orbit by the proof of [Sou94, Proposition 7]. When C is reducible, C must have two irreducible components, C^+ and C^-, and the computation from Example 2 shows that the ϕ-stable sheaves are either the line bundles of bidegree $(g^+ - 1, g^-)$ or the line bundles of bidegree $(g^+, g^- - 1)$. In the first case, restricting to components defines an isomorphism $\overline{\mathcal{J}}_C(\phi) \cong \mathcal{J}^{g^+-1}_{C^+} \times \mathcal{J}^{g^-}_{C^-}$ that identifies $\overline{\Theta}(\phi)$ with $p_2^*(\text{node} + \overline{\Theta}_{C^+}) + p_2^*(\overline{\Theta}_{C^-})$. (Here p_1, p_2 are the projection morphisms). This identifies $(\overline{\mathcal{J}}_C(\phi), \overline{\Theta}_C)$ as the product of principally polarized varieties, and such a product satisfies the desired conditions. The case of bidegree $(g^+, g^- - 1)$ is entirely analogous, with the roles of C^+ and C^- being switched. □

Remark 12. Observe that Lemma 24 implies that the unique extension of $(\mathcal{J}_{g,n}, \overline{\Theta})$ to a family of stable pairs over $\mathcal{U}$ admits multiple descriptions as a moduli space. The authors expect this remains true over $\mathcal{M}^{(0)}_{g,n}$ but, as our goal is to establish (8), we do not pursue this issue here.

An immediate consequence is

Corollary 25. *Equation* (8) *holds.*

Proof. By Lemma 24 $[\overline{D}_{\vec{d}}(\text{SP})] = [\overline{D}_{\vec{d}}(\phi)]$ for any ϕ satisfying the conditions from Lemma 16. The other equalities follow from Hain's result (47) and Theorem 23. □

5.3. **The class of Müller.** Müller studied a different extension of $[D_{\vec{d}}]$ in [Mül13]. Under the assumption that some d_j is negative, he defined $\overline{D}_{\vec{d}}(\text{Mü}) \subset \overline{\mathcal{M}}_{g,n}$ to be the

Zariski closure of $D_{\vec{d}}$ and then computed

$$(49)\quad [\overline{D}_{\vec{d}}(\text{Mü})] = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \cdot \psi_j - \sum_{\substack{i,S \\ S \subseteq S^+}} \binom{|d_S - i| + 1}{2} \cdot \delta_{i,S} - \sum_{\substack{i,S \\ S \not\subseteq S^+}} \binom{d_S - i + 1}{2} \cdot \delta_{i,S}$$

in [Mül13, Theorem 5.6]. (Here $S^+ := \{j \in \{1, \dots, n\} : d_j > 0\}$). Grushevsky-Zakharov gave an alternative proof of this in [GZ14, Theorem 2].

Comparing (42) with (49), we see that if ϕ_0 is as in Lemma 16, then

$$(50)\qquad [\overline{D}_{\vec{d}}(\phi_0)] = [\overline{D}_{\vec{d}}(\text{Mü})] + \sum_{(i,S) \in T_{\vec{d}}} (i - d_S) \cdot \delta_{i,S},$$

where $T_{\vec{d}}$ is defined by

$$T_{\vec{d}} := \{(i,S) \colon\ d_j > 0 \text{ for all } j \in S, \text{ and } d_S < i\}.$$

Inspecting Equation (50), we see that the divisor classes $[\overline{D}_{\vec{d}}(\phi_0)]$ and $[\overline{D}_{\vec{d}}(\text{Mü})]$ are equal if and only if $T_{\vec{d}} = \varnothing$. Thus from Lemma 20 we deduce the following.

Corollary 26. *The inclusion of the closed substack $\overline{D}_{\vec{d}}(M\ddot{u})$ in $\overline{D}_{\vec{d}}(\phi)$ is an isomorphism if and only if $\phi = \phi_0$ and $T_{\vec{d}} = \varnothing$.*

While $[\overline{D}_{\vec{d}}(\text{Mü})]$ is not always equal to some $[\overline{D}_{\vec{d}}(\phi)]$, it is always equal to a divisor class on $\overline{\mathcal{J}}_{g,n}(\phi)$: the difference $[\overline{D}_{\vec{d}}(\phi_0)] - \sum (i - d_S) \cdot \delta_{i,S}$. However, this divisor class can be chosen to be an effective divisor only when the hypotheses of Corollary 26 hold.

Proposition 27. *The class $[\overline{D}_{\vec{d}}(M\ddot{u})]$ equals the pullback via $s_{\vec{d}}$ of an effective divisor on $\overline{\mathcal{J}}_{g,n}(\phi)$ if and only if it equals $[\overline{D}_{\vec{d}}(\phi_0)]$.*

Proof. The pullback homomorphism $s^*_{\vec{d}} \colon \text{Pic}(\overline{\mathcal{J}}_{g,n}(\phi)) \to \text{Pic}(\mathcal{M}^{(0)}_{g,n})$ is injective because the image of each boundary class is a boundary class, and $\overline{\theta}(\phi)$ is not the linear combination of boundary classes. Moreover, because the Picard groups of $\overline{\mathcal{J}}_{g,n}(\phi)$ are all isomorphic, it is enough to consider the case $\phi = \phi_0$. Therefore, if this effective divisor exists it has to be linearly equivalent to the difference

$$\overline{\theta}(\phi_0) - \sum_{(i,S) \in T_{\vec{d}}} (i - d_S) \cdot \delta_{i,S}.$$

The proof is then concluded by applying Lemma 28. □

Lemma 28. *The only effective divisor linearly equivalent to $\overline{\Theta}(\phi)$ is $\overline{\Theta}(\phi)$ itself.*

Proof. It is enough to prove that the space of global sections of $\mathcal{O}(\overline{\Theta}(\phi))$ is 1-dimensional, and we prove this by computing the direct image of $\mathcal{O}(\overline{\Theta}(\phi))$ under the natural morphism $\overline{\mathcal{J}}_{g,n}(\phi) \to \mathcal{M}^{(0)}_{g,n}$. We compute this direct image by using the theorem on cohomology and base change together with results about the theta divisor of a Jacobian variety.

Let $\mathcal{U} \subset \mathcal{M}^{(0)}_{g,n}$ be the open substack parameterizing marked curves with at most one node. Given $(C, p_1, \dots, p_n) \in \mathcal{U}$, consider the restriction $\mathcal{O}(\overline{\Theta}_C(\phi))$ of $\mathcal{O}(\overline{\Theta}(\phi))$ to the fiber J_C over C. The authors claim that the space of global sections of this restriction is 1-dimensional. In the proof of Lemma 24, we observe that when C is of compact type $(\overline{J}_C, \overline{\Theta}_C)$ is a product of principally polarized Jacobians, so the result follows from the Riemann–Roch formula for abelian varieties [Mum08, page 140]. When C is not of compact type, $(\overline{J}_C, \overline{\Theta}_C(\phi))$ is a principally polarized rank 1 degeneration (in the sense of [Mum83]), and the desired computation is done on [Mum83, page 4] (by reducing the claim to a result about the normalization of $\overline{\mathcal{J}}_C$, which is a $\mathbb{P}^1$-bundle over an abelian variety).

Now consider the global section of $\mathcal{O}(\overline{\Theta}(\phi))|\mathcal{U}$ that is the image of 1 under the natural inclusion $\mathcal{O}_{\mathcal{U}} \to \mathcal{O}(\overline{\Theta}(\phi))|\mathcal{U}$. This section has nonzero restriction to every fiber of $\overline{\mathcal{J}}_{g,n}(\phi)|\mathcal{U} \to \mathcal{U}$ (because $\overline{\Theta}(\phi)$ does not contain a fiber), so by dimension considerations, this restriction is a generator. We can conclude that the hypothesis of the cohomology and base change theorem [Ill05, Theorem 8.3.2] is satisfied, so the formation of the direct image of $\mathcal{O}(\overline{\Theta}(\phi))|\mathcal{U}$ under $\overline{\mathcal{J}}_{g,n}(\phi)|\mathcal{U} \to \mathcal{U}$ commutes with base change. In particular, the natural inclusion from $\mathcal{O}_{\mathcal{U}}$ to the direct image has the property that the induced map on stalks is always surjective. We conclude that this inclusion is an isomorphism, so

$$\begin{aligned} h^0(\overline{\mathcal{J}}_{g,n}(\phi)|\mathcal{U}, \mathcal{O}(\overline{\Theta}(\phi))|\mathcal{U}) &= h^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \\ &= 1. \end{aligned}$$

The last equality follows because $\mathcal{O}_{\mathcal{U}}$ satisfies condition S2 and the complement of $\mathcal{U}$ has codimension 2. For similar reasons,

$$h^0(\overline{\mathcal{J}}_{g,n}(\phi), \mathcal{O}(\overline{\Theta}(\phi))) = h^0(\overline{\mathcal{J}}_{g,n}(\phi), \mathcal{O}(\overline{\Theta}(\phi))|\mathcal{U}).$$

□

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References

[ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli, *Twisted bundles and admissible covers*, Comm. Algebra **31** (2003), no. 8, 3547–3618, Special issue in honor of Steven L. Kleiman. MR 2007376 (2005b:14049)

[Ale02] Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708. MR 1923963 (2003g:14059)

[Ale04] ———, *Compactified Jacobians and Torelli map*, Publ. Res. Inst. Math. Sci. **40** (2004), no. 4, 1241–1265. MR 2105707 (2006a:14016)

[Bea77] Arnaud Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), no. 2, 149–196. MR 0572974 (58 #27995)

[Bin05] Gilberto Bini, *Chern classes of the moduli stack of curves*, Math. Res. Lett. **12** (2005), no. 5-6, 759–766. MR 2189236 (2006j:14034)

[Cap94] Lucia Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), no. 3, 589–660. MR 1254134 (95d:14014)

[Cap08a] ———, *Compactified Jacobians, Abel maps and theta divisors*, Curves and abelian varieties, Contemp. Math., vol. 465, Amer. Math. Soc., Providence, RI, 2008, pp. 1–23. MR 2457733 (2010b:14088)

[Cap08b] ———, *Néron models and compactified Picard schemes over the moduli stack of stable curves*, Amer. J. Math. **130** (2008), no. 1, 1–47. MR 2382140 (2009j:14030)

[Cap09] ———, *Geometry of the theta divisor of a compactified Jacobian*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 6, 1385–1427. MR 2557139 (2011c:14093)

[CMKV12] Sebastian Casalaina-Martin, Jesse Leo Kass, and Filippo Viviani, *The local structure of compactified Jacobians*, http://arxiv.org/abs/1107.4166, 2012.

[Dud15] Bashar Dudin, *Compactified universal jacobian and the double ramification cycle*, arXiv:1505.02897, 2015.

[Edi13] Dan Edidin, *Riemann-Roch for Deligne-Mumford stacks*, A celebration of algebraic geometry, Clay Math. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 2013, pp. 241–266. MR 3114943

[EGS13] Dan Edidin, Anton Geraschenko, and Matthew Satriano, *There is no degree map for 0-cycles on Artin stacks*, Transform. Groups **18** (2013), no. 2, 385–389. MR 3055771

[Est01] Eduardo Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3045–3095 (electronic). MR 1828599 (2003b:14036)

[Fer03] Daniel Ferrand, *Conducteur, descente et pincement*, Bull. Soc. Math. France **131** (2003), no. 4, 553–585. MR 2044495 (2005a:13016)

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)

[GZ14] Samuel Grushevsky and Dmitry Zakharov, *The double ramification cycle and the theta divisor*, Proc. Amer. Math. Soc. **142** (2014), no. 12, 4053–4064. MR 3266977

[Hai13] Richard Hain, *Normal functions and the geometry of moduli spaces of curves*, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, pp. 527–578. MR 3184171

[Ill05] Luc Illusie, *Grothendieck's existence theorem in formal geometry*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, With a letter (in French) of Jean-Pierre Serre, pp. 179–233. MR 2223409

[Jar00] Tyler J. Jarvis, *Compactification of the universal Picard over the moduli of stable curves*, Math. Z. **235** (2000), no. 1, 123–149. MR 1785075 (2001m:14040)

[Knu02] Finn F. Knudsen, *Determinant functors on exact categories and their extensions to categories of bounded complexes*, Michigan Math. J. **50** (2002), no. 2, 407–444. MR 1914072 (2003k:18015a)

[Mel09] Margarida Melo, *Compactified Picard stacks over $\overline{\mathcal{M}}_g$*, Math. Z. **263** (2009), no. 4, 939–957. MR 2551606 (2010j:14018)

[Mel11] ———, *Compactified Picard stacks over the moduli stack of stable curves with marked points*, Adv. Math. **226** (2011), no. 1, 727–763. MR 2735773 (2011j:14028)

[MRV14] Margarida Melo, Antonio Rapagnetta, and Filippo Viviani, *Fine compactified Jacobians of reduced curves*, arXiv:1406.2299, 2014.

[Mül13] Fabian Müller, *The pullback of a theta divisor to $\overline{\mathcal{M}}_{g,n}$*, Math. Nachr. **286** (2013), no. 11-12, 1255–1266. MR 3092284

[Mum83] David Mumford, *On the Kodaira dimension of the Siegel modular variety*, Algebraic geometry—open problems (Ravello, 1982), Lecture Notes in Math., vol. 997, Springer, Berlin, 1983, pp. 348–375. MR 714757 (85d:14061)

[Mum08] ———, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008, With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. MR 2514037 (2010e:14040)

[OS79] Tadao Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90. MR 536936 (82e:14054)

[Pan96] Rahul Pandharipande, *A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles*, J. Amer. Math. Soc. **9** (1996), no. 2, 425–471. MR 1308406 (96f:14014)

[Sim94] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 47–129. MR 1307297 (96e:14012)

[Sou94] A. Soucaris, *The ampleness of the theta divisor on the compactified Jacobian of a proper and integral curve*, Compositio Math. **93** (1994), no. 3, 231–242. MR 1300762 (95m:14017)

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